ASKEY-WILSON POLYNOMIALS, QUADRATIC HARNESSES AND MARTINGALES

WŁODEK BRYC AND JACEK WESOŁOWSKI

Abstract. We use orthogonality measures of Askey-Wilson polynomials to construct Markov processes with linear regressions and quadratic conditional variances. Askey-Wilson polynomials are orthogonal martingale polynomials for these processes.

1. Introduction

Orthogonal martingale polynomials for stochastic processes have been studied by a number of authors, see [5, 13, 18, 22, 24, 25, 27, 28]. Orthogonal martingale polynomials play also a prominent role in non-commutative probability [1, 2], and can serve as a connection to the so called "classical versions" of non-commutative processes. On the other hand, classical versions may exist without polynomial martingale structure, see [6]. In [8] we identify intrinsic properties of the first two conditional moments of a process which guarantee that a stochastic process has orthogonal martingale polynomials. These properties, linear conditional expectations and quadratic conditional variances, which we call the quadratic harness properties, already lead to a number of new examples of Markov processes [9, 10, 11] with orthogonal martingale polynomials. Random fields with harness property were introduced by Hammersley [14] and their properties were studied e. g. in [19, 32].

In this paper we use measures of orthogonality of Askey-Wilson polynomials to construct a large class of Markov processes with quadratic harness property that includes previous examples either as special cases or as "boundary cases". The main step is the construction of an auxiliary Markov process which has Askey-Wilson polynomials [4] as orthogonal martingale polynomials. The question of probabilistic interpretation of Askey-Wilson polynomials was raised in [12, page 197].

The paper is organized as follows. In the reminder of this section we recall background material on quadratic harness property, Askey-Wilson polynomials, we state our two main results, and conclude with some applications. In Section 2 we give an elementary construction that does not cover the entire range of parameters, but it is explicit and does not rely on orthogonal polynomials. The general construction appears in Section 3; this proof follows the method from [9] and relies on martingale property of Askey-Wilson polynomials which extends projection formula for Askey-Wilson polynomials [20] to a large range of parameters. Section 4

Date: December 3, 2008 .
2000 *Mathematics Subject Classification.* Primary: 60J25; Secondary: 46L53 .
Key words and phrases. Quadratic conditional variances, harnesses, orthogonal martingale polynomials, free Lévy processes, hypergeometric orthogonal polynomials.

contains another elementary but computationally more cumbersome construction of a purely discrete quadratic harness which is not covered by Theorem 1.1.

1.1. **Quadratic harnesses.** In [8] the authors consider square-integrable stochastic processes on $[0, \infty)$ such that for all $t, s \geq 0$,

$$\mathbb{E}(X_t) = 0,\ \mathbb{E}(X_t X_s) = \min\{t, s\}, \tag{1.1}$$

$E(X_t|\mathcal{F}_{s,u})$ is a linear function of X_s, X_u, and $\mathrm{Var}[X_t|\mathcal{F}_{s,u}]$ is a quadratic function of X_s, X_u. Here, $\mathcal{F}_{s,u}$ is the two-sided σ-field generated by $\{X_r : r \in (0, s) \cup (u, \infty)\}$. We will also use the one sided σ-fields $\mathcal{F}_t$ generated by $\{X_t : r \leq t\}$.

Then for all $s < t < u$,

$$\mathbb{E}(X_t|\mathcal{F}_{s,u}) = \frac{u - t}{u - s} X_s + \frac{t - s}{u - s} X_u, \tag{1.2}$$

and under certain technical assumptions, [8, Theorem 2.2] asserts that there exist numerical constants $\eta, \theta, \sigma, \tau, \gamma$ such that for all $s < t < u$,

$$\begin{aligned} &\mathrm{Var}[X_t|\mathcal{F}_{s,u}] \\ &= \frac{(u - t)(t - s)}{u(1 + \sigma s) + \tau - \gamma s} \left(1 + \sigma \frac{(uX_s - sX_u)^2}{(u - s)^2} + \eta \frac{uX_s - sX_u}{u - s} \right. \\ &\quad \left. + \tau \frac{(X_u - X_s)^2}{(u - s)^2} + \theta \frac{X_u - X_s}{u - s} - (1 - \gamma) \frac{(X_u - X_s)(uX_s - sX_u)}{(u - s)^2} \right). \end{aligned} \tag{1.3}$$

In this paper it will be convenient to extend [8] by admitting also processes defined on a time domain T which is a proper subset of $[0, \infty)$, and by relaxing slightly condition (1.1). (Compare [8, Proposition 2.1].) Accordingly, we will say that a square-integrable stochastic process $(X_t)_{t \in T}$ is a quadratic harness on T with parameters $(\eta, \theta, \sigma, \tau, \gamma)$, if it satisfies (1.2) and (1.3) on T.

Under the conditions listed in [8, Theorem 2.4 and Theorem 4.1], quadratic harnesses on $(0, \infty)$ have orthogonal martingale polynomials. Although several explicit three step recurrences have been worked out in [8, Section 4] and for some of them corresponding quadratic harnesses were constructed in a series of papers [9, 10, 11], the general orthogonal martingale polynomials have not been identified, and the question of existence of corresponding quadratic harnesses was left open.

It has been noted that the family of all quadratic harnesses on $(0, \infty)$ that satisfy condition (1.1) is invariant under the action of translations and reflections of $\mathbb{R}$: translation by $a \in \mathbb{R}$ acts as $(X_t) \mapsto e^{-a} X_{e^{2a} t}$ and the reflection at 0 acts as $(X_t) \mapsto (tX_{1/t})$. Since translations and reflection generate also the symmetry group of the Askey-Wilson polynomials [21], it is natural to investigate how to relate the measures of orthogonality of the Askey-Wilson polynomials to quadratic harnesses. The goal of this paper is to explore this idea and significantly enlarge the class of available examples. We show that quadratic harnesses exist and are Markov processes for a wide range of parameters $(\eta, \theta, \sigma, \tau, \gamma)$. The basic Markov process we construct has Askey-Wilson polynomials as orthogonal martingale polynomials. The final quadratic harness is then obtained by appropriate scaling and deterministic change of time.

Since Askey-Wilson polynomials together with their limits are regarded as the most general classical polynomials, it is natural to inquire whether all quadratic harnesses can be obtained by taking limits of processes with Askey-Wilson transition probabilities.

Theorem 1.1. *Fix parameters $-1 < q < 1$ and A, B, C, D that are either real or (A, B) or (C, D) are complex conjugate pairs, and such that $ABCD, qABCD < 1$. Assume that*

$$AC, AD, BC, BD, qAC, qAD, qBD, qBD \in \mathbb{C} \setminus [1, \infty). \tag{1.4}$$

Let

$$\eta = -\frac{((A+B)(1+ABCD) - 2AB(C+D))\sqrt{1-q}}{\sqrt{(1-AC)(1-BC)(1-AD)(1-BD)(1-qABCD)}}, \tag{1.5}$$

$$\theta = -\frac{((D+C)(1+ABCD) - 2CD(A+B))\sqrt{1-q}}{\sqrt{(1-AC)(1-BC)(1-AD)(1-BD)(1-qABCD)}}, \tag{1.6}$$

$$\sigma = \frac{AB(1-q)}{1-qABCD}, \tag{1.7}$$

$$\tau = \frac{CD(1-q)}{1-qABCD}, \tag{1.8}$$

$$\gamma = \frac{q-ABCD}{1-qABCD}. \tag{1.9}$$

Consider the interval $J = (T_0, T_1)$ of positive length defined by

$$T_0 = \max\left\{0, \frac{\gamma - 1 + \sqrt{(\gamma-1)^2 - 4\sigma\tau}}{2\sigma}, -\tau\right\}, \tag{1.10}$$

$$\frac{1}{T_1} = \max\left\{0, \frac{\gamma - 1 + \sqrt{(\gamma-1)^2 - 4\sigma\tau}}{2\tau}, -\sigma\right\}. \tag{1.11}$$

Then there exists a bounded Markov process $(X_t)_{t\in J}$ with mean and covariance (1.1) *such that* (1.2) *holds, and* (1.3) *holds with parameters $\eta, \theta, \sigma, \tau, \gamma$. Process $(X_t)_{t\in J}$ is unique among the processes with infinitely-supported one-dimensional distributions that have moments of all orders and satisfy* (1.1)*,* (1.2)*, and* (1.3) *with the same parameters $\eta, \theta, \sigma, \tau, \gamma$.*

The assumptions on A, B, C, D are dictated by the desire to limit the number of cases in the proof but do not exhaust all possibilities where quadratic harness (X_t) with Askey-Wilson transition probabilities exists; see Proposition 4.1.

Remark 1.1. In Theorem 1.1, $(\gamma-1)^2 > 4\sigma\tau$. To see this, write $(\gamma-1)^2 - 4\sigma\tau = (1-ABCD)^2(1-q)^2/(1-qABCD)^2$. Together with the upper bound in [8, Theorem 2.2] this shows that to construct a quadratic harness on $[0, \infty)$ from Theorem 1.1, we must have $-1 < \gamma < 1 - 2\sqrt{\sigma\tau}$.

Remark 1.2. In terms of the original parameters, the end-points of the interval are:

$$T_0 = \max\left\{0, -CD, \frac{-CD(1-q)}{1-qABCD}\right\}, \tag{1.12}$$

$$\frac{1}{T_1} = \max\left\{0, -AB, \frac{-AB(1-q)}{1-qABCD}\right\}. \tag{1.13}$$

If $CD < 0$ or $AB < 0$ then the third term under the maximum contributes for $q < 0$ only.

Remark 1.3. From (1.9), the construction will give $\gamma > 1$ when $ABCD < -1$. Multiplying (1.7) and (1.8), we see that this may occur only when $\sigma\tau < 0$, i.e. for harnesses on finite intervals; compare [8, Theorem 2.2].

1.2. **Martingale property of Askey-Wilson polynomials.** For $a, b, c, d \in \mathbb{C}$ such that $a \neq 0$, and

$$abcd, qabcd, ab, qab, ac, qac, ad, qad \in \mathbb{C} \setminus [1, \infty), \tag{1.14}$$

the Askey-Wilson [4] polynomials $\bar{w}_n(x) = \bar{w}_n(x; a, b, c, d)$ are defined by the recurrence

$$(2x - (a + a^{-1}))\bar{w}_n(x) = A_n\bar{w}_{n+1}(x) - (A_n + C_n)\bar{w}_n + C_n\bar{w}_{n-1}(x), \; n \geq 0, \tag{1.15}$$

where

$$A_n = \frac{(1 - abcdq^{n-1})(1 - abq^n)(1 - acq^n)(1 - adq^n)}{a(1 - abcdq^{2n-1})(1 - abcdq^{2n})}, \tag{1.16}$$

$$C_n = \frac{a(1 - q^n)(1 - bcq^{n-1})(1 - bdq^{n-1})(1 - cdq^{n-1})}{(1 - abcdq^{2n-2})(1 - abcdq^{2n-1})}. \tag{1.17}$$

The initial conditions are $\bar{w}_{-1} = 0$ and $\bar{w}_0 = 1$. We also note that recurrence (1.15) is symmetric in parameters b, c, d but a is distinguished.

Askey-Wilson polynomials $\bar{w}_n(x; a, b, c, d)$ can be defined also for $a = 0$; the definition appropriate for our purposes uses the limit as $a \to 0$ of the normalized (monic) form of the three step recurrence:

$$xw_n(x) = w_{n+1}(x) - \frac{1}{2}(A_n + C_n - (a + a^{-1}))w_n + \frac{1}{4}A_{n-1}C_n w_{n-1}(x). \tag{1.18}$$

Recall that polynomials $\{r_n(x; t) : n \in \mathbb{Z}_+, t \in I\}$ are orthogonal martingale polynomials for process $(Z_t)_{t \in I}$ if:

(i) $\mathbb{E}\,(r_n(Z_t; t)r_m(Z_t; t)) = 0$ for $m \neq n$ and $t \in I$.
(ii) $\mathbb{E}\,(r_n(Z_t; t)|\mathcal{F}_s) = r_n(Z_s; s)$ for $s < t$ in I and all $n = 0, 1, 2 \ldots$.

The following result shows that Askey-Wilson polynomials are orthogonal martingale polynomials for a family of Markov processes. If A, B, C, D are as in Proposition 2.1, this is a re-interpretation of the projection formula from [20].

Theorem 1.2. *Suppose that A, B, C, D satisfy the assumptions in Theorem 1.1. Let*

$$I = I(A, B, C, D, q) = \left(\max\{0, CD, qCD\}, \frac{1}{\max\{0, AB, qAB\}}\right) \tag{1.19}$$

with the convention $1/0 = \infty$. (The last terms under the maxima can contribute only when $q < 0$ and CD or AB are negative.) If $A \neq 0$, let

$$r_n(x; t) = \bar{w}_n\left(\frac{\sqrt{1-q}}{2\sqrt{t}}\, x; A\sqrt{t}, B\sqrt{t}, C/\sqrt{t}, D/\sqrt{t}\right). \tag{1.20}$$

(If $A = 0$, use monic w_n instead of $\bar{w}_n$.) Then

$$\{r_n(x; t) : n = 0, 1, 2 \ldots, t \in I\}$$

are orthogonal martingale polynomials for a Markov process (Z_t) which satisfies (1.2) and (1.3) with $\eta = \theta = \sigma = \tau = 0$ and $\gamma = q$.

Remark 1.4. Process (Z_t) from Theorem 1.2 is closely related to the Markov process (X_t) that appears in the statement of Theorem 1.1, see (2.27).

Remark 1.5. If $A, B \neq 0$ we can choose either A or B as the distinguished first parameter in $\bar{w}_n$. The corresponding polynomials r_n differ by the factor $\frac{A^n(BC,BD)_n}{B^n(AC,AD)_n}$ which does not depend on t, so both sets of polynomials are orthogonal martingale polynomials for the same process (Z_t).

1.3. **Some worked out examples.** This section shows how Theorem 1.1 is related to some previous constructions, and how it yields new examples. From examples that have been previously worked out in detail one can see that the boundary of the range of parameters is not covered by Theorem 1.1; in particular it does not cover at all the family of five Meixner Lévy processes characterized by quadratic harness property in [31]. On the other hand, sometimes new examples arise when processes run only on a subinterval of $(0, \infty)$.

1.3.1. *q-Meixner processes.* Theorem 1.1 allows us to extend [10, Theorem 3.5] to negative τ. (The cases $\gamma = \pm 1$ which are included in [10] are not covered by Theorem 1.1.)

Corollary 1.3. *Fix $\tau, \theta \in \mathbb{R}$ and $-1 < \gamma < 1$ and let*

$$T_0 = \begin{cases} 0 & \text{if } \tau \geq 0 \\ -\tau/(1-\gamma) & \text{if } \tau < 0,\ \gamma \geq 0 \\ -\tau & \text{if } \tau < 0,\ \gamma < 0. \end{cases}$$

Then there exists a Markov process (X_t) on $[T_0, \infty)$ such that (1.1), (1.2) *hold, and* (1.3) *holds with parameters $\eta = 0, \sigma = 0$.*

Proof. Let $q = \gamma$, $A = 0$, $B = 0$, and

$$C = \begin{cases} \frac{-\theta+\sqrt{\theta^2-4\tau}}{2\sqrt{1-q}} & \theta^2 \geq 4\tau \\ \frac{-\theta+i\sqrt{4\tau-\theta^2}}{2\sqrt{1-q}} & \theta^2 < 4\tau \end{cases}, \quad D = \begin{cases} \frac{-\theta-\sqrt{\theta^2-4\tau}}{2\sqrt{1-q}} & \theta^2 \geq 4\tau \\ \frac{-\theta-i\sqrt{4\tau-\theta^2}}{2\sqrt{1-q}} & \theta^2 < 4\tau \end{cases}$$

Then (1.4) holds trivially so by Theorem 1.1, (X_t) is well defined on $J = (T_0, \infty)$. Straightforward calculation of the parameters from (1.6), (1.8), (1.9) ends the proof. ☐

When $\tau < 0$, the univariate laws of X_t are the "sixth" family to be added to the five cases from [10, Theorem 3.5]. The polynomials orthogonal with respect to the law of X_t satisfy recurrence

$$xp_n(x;t) = p_{n+1}(x;t) + \theta[n]_q p_n(x;t) + (t + \tau[n-1]_q)[n]_q p_{n-1}(x;t),$$

where $[n]_q = (1-q^n)/(1-q)$. So the polynomials orthogonal with respect to the standardized law of $X_t/\sqrt{t}$ are

$$x\tilde{p}_n(x;t) = \tilde{p}_{n+1}(x;t) + \frac{\theta}{\sqrt{t}}[n]_q\tilde{p}_n(x;t) + (1 + \frac{\tau}{t}[n-1]_q)[n]_q\tilde{p}_{n-1}(x;t). \tag{1.21}$$

The same law appears under the name q-Binomial law in [23] for parameters $n = -t/\tau \in \mathbb{N}$, $\tau = -p(1-p) \in [-1/4, 0)$. When $q \leq 0$, this law is discrete two-point law at $t = |\tau|$.

A justification of relating this law to the Binomial can be given for $q = 0$. In this case, recurrence (1.21) appears in [7, (3)] with their $a = \frac{\theta}{\sqrt{t}}$ and their $b = \frac{\tau}{t}$. By [7, Proposition 2.1], the law ν_t of $\frac{1}{\sqrt{\tau}}X_t$ is a free convolution $\frac{t}{|\tau|}$-fold power of the two-point discrete law that corresponds to $t = -\tau$. That is, $\nu_t = \nu_{-\tau}^{t/|\tau|\boxplus}$; in particular,

at $t=-n\tau$, $X_t/\sqrt{\tau}$ has the law that is the n-fold free additive convolution of a centered and standardized two-point law.

1.3.2. *Bi-Poisson processes.* Next we deduce a version of [9, Theorem 1.2]. Here we again have to exclude the boundary cases $\gamma=\pm 1$ as well as the case $1+\eta\theta=\max\{\gamma,0\}$.

Corollary 1.4. *For $-1<\gamma<1$, and $1+\eta\theta>\max\{\gamma,0\}$ there exists a Markov process $(X_t)_{t\in[0,\infty)}$ such that* (1.1), (1.2) *hold, and* (1.3) *holds with $\sigma=\tau=0$.*

Proof. Let $A=0$, $B=-\frac{\eta}{\sqrt{\eta\theta+1-q}}$, $C=0$, $D=-\frac{\theta}{\sqrt{\eta\theta+1-q}}$. Then $BD=\frac{\eta\theta}{\eta\theta+1-q}<1$. The condition $qBD<1$ is also satisfied, as we assume $\eta\theta+1>0$ when $q<0$. Thus (1.4) holds and we can apply Theorem 1.1. From formulas (1.5) through (1.9), the quadratic harness has parameters $\eta,\theta,\sigma=0,\tau=0,\gamma$, as claimed. □

1.3.3. *Free harness.* Next we indicate the range of parameters that guarantee existence of the processes described in [8, Proposition 4.3]. Let

$$\alpha=\frac{\eta+\theta\sigma}{1-\sigma\tau},\ \beta=\frac{\eta\tau+\theta}{1-\sigma\tau}. \tag{1.22}$$

Corollary 1.5. *For $0\leq\sigma\tau<1$, $\gamma=-\sigma\tau$, and η,θ with $2+\eta\theta+2\sigma\tau\geq 0$ and $1+\alpha\beta>0$, there exists a Markov process $(X_t)_{t\in[0,\infty)}$ such that* (1.1), (1.2) *and* (1.3) *hold.*

Remark 1.6. When $2+\eta\theta+2\sigma\tau<0$, two of the products in (1.4) are in the "forbidden region" $[1,\infty)$, so Theorem 1.1 does not apply. However, the univariate Askey-Wilson distributions are still well defined.

Proof of Corollary 1.5. Take $q=0$, and let

$$A=-\frac{\alpha+\beta\,\sigma-\sqrt{-4\,\sigma+(\alpha-\beta\,\sigma)^2}}{2\sqrt{1+\alpha\,\beta}},\ B=-\frac{\alpha+\beta\,\sigma+\sqrt{-4\,\sigma+(\alpha-\beta\,\sigma)^2}}{2\sqrt{1+\alpha\,\beta}},$$

$$C=-\frac{\beta+\alpha\,\tau-\sqrt{-4\,\tau+(\beta-\alpha\,\tau)^2}}{2\sqrt{1+\alpha\,\beta}},\ D=-\frac{\beta+\alpha\,\tau+\sqrt{-4\,\tau+(\beta-\alpha\,\tau)^2}}{2\sqrt{1+\alpha\,\beta}}.$$

To verify that $AC\notin[1,\infty)$ we proceed as follows. Note that

$$A+B = -\frac{\alpha+\sigma\beta}{\sqrt{1+\alpha\beta}}, \tag{1.23}$$

$$C+D = -\frac{\alpha\tau+\beta}{\sqrt{1+\alpha\beta}}, \tag{1.24}$$

$$A-B = \frac{\sqrt{(\alpha-\sigma\beta)^2-4\sigma}}{\sqrt{1+\alpha\beta}}, \tag{1.25}$$

$$C-D = \frac{\sqrt{(\beta-\tau\alpha)^2-4\tau}}{\sqrt{1+\alpha\beta}}. \tag{1.26}$$

Multiplying $(A+B)(C+D)$ and $(A-B)(C-D)$ and using $ABCD=\sigma\tau$ we get

$$AC+\frac{\sigma\tau}{AC}-BC-\frac{\sigma\tau}{BC}=\frac{\sqrt{(\alpha-\sigma\beta)^2-4\sigma}\sqrt{(\beta-\tau\alpha)^2-4\tau}}{1+\alpha\beta}$$

and

$$AC+\frac{\sigma\tau}{AC}+BC+\frac{\sigma\tau}{BC}=\frac{(\alpha+\sigma\beta)(\alpha\tau+\beta)}{1+\alpha\beta}.$$

This gives the following quadratic equation for AC:

$$AC + \frac{\sigma\tau}{AC} = \frac{(\alpha+\sigma\beta)(\alpha\tau+\beta) + \sqrt{(\alpha-\sigma\beta)^2-4\sigma}\sqrt{(\beta-\tau\alpha)^2-4\tau}}{2(1+\alpha\beta)}. \tag{1.27}$$

We now note that a quadratic equation $x + a/x = b$ with $0 < a < 1$ and complex b can have a root in $[1, \infty)$ only when b is real and $b \geq 1 + a$; this follows from the fact that $x + a/x$ is increasing for $x > a$, so $x + a/x \geq 1 + a$ for $x \geq 1$.

Suppose therefore that the right hand side of (1.27) is real and larger that $1+\sigma\tau$. Then calculations lead to $\sqrt{\eta^2-4\sigma}\sqrt{\theta^2-4\tau} \geq 2+\eta\theta+2\sigma\tau$. The right hand side is nonnegative by assumption, so squaring the inequality we get $(1+\alpha\,\beta)\,(1-\sigma\,\tau)^2 \leq 0$, which contradicts the assumption.

Other cases with AD, BC, BD are handled similarly. Since $ABCD = \sigma\tau < 1$ by assumption, by Theorem 1.1 the quadratic harness exists.

It remains to calculate the parameters. From $AB = \sigma$ $CD = \tau$ we see that (1.7) and (1.8) give the correct values, and $\gamma = -\sigma\tau$ from (1.9). To compute the remaining parameters, we re-write the expression under the square root in the denominator of (1.5) as

$$\begin{aligned}(1-AC)&(1-BC)(1-AD)(1-BD)\\ &= \left(1+\sigma\tau-(AC+\frac{\sigma\tau}{AC})\right)\left(1+\sigma\tau-(BC+\frac{\sigma\tau}{BC})\right).\end{aligned}$$

This is the product of two conjugate expressions (see (1.27), and its derivation). A calculation now simplifies the denominator of (1.5) to $(1-\sigma\tau)/\sqrt{1+\alpha\beta}$. Inserting (1.23) and (1.24), the numerator of (1.5) simplifies to $(\alpha-\beta\sigma)\,(1-\sigma\tau)/\sqrt{1+\alpha\beta}$ The quotient of these two expressions is $\alpha - \beta\sigma = \eta$. Similar calculation verifies (1.6). ☐

1.3.4. *Purely quadratic harness.* The quadratic harness with parameters $\eta = \theta = 0$, and $\sigma\tau > 0$ has not been previously constructed.

Corollary 1.6. *For $\sigma, \tau > 0$ with $\sigma\tau < 1$ and $-1 < \gamma < 1 - 2\sqrt{\sigma\tau}$ there exists a Markov process $(X_t)_{t\in[0,\infty)}$ such that* (1.1), (1.2) *hold, and* (1.3) *holds with $\eta = \theta = 0$.*

Proof. Let

$$q = \frac{4(\gamma+\sigma\tau)}{\left(1+\gamma+\sqrt{(1-\gamma)^2-4\sigma\tau}\right)^2}.$$

To see that $-1 < q < 1$, note that for $\gamma + \sigma\tau \neq 0$,

$$q = \frac{1+\gamma^2-2\,\sigma\,\tau-(1+\gamma)\sqrt{(1-\gamma)^2-4\,\sigma\,\tau}}{2\,(\gamma+\sigma\,\tau)},$$

which gives

$$q - 1 = \frac{-2\sqrt{(1-\gamma)^2-4\,\sigma\,\tau}}{1+\gamma+\sqrt{(1-\gamma)^2-4\sigma\tau}} < 0\,, \tag{1.28}$$

and

$$q+1=\frac{2\,(1+\gamma)}{1+\gamma+\sqrt{(1-\gamma)^2-4\,\sigma\,\tau}}>0\,. \tag{1.29}$$

Noting that $(1-q)^2+4q\sigma\tau \geq 4\sigma\tau(1-\sigma\tau)>0$, let

$$A=-B=\frac{i\sqrt{2\sigma}}{\sqrt{(1-q)+\sqrt{(1-q)^2+4\,q\,\sigma\,\tau}}}$$

and

$$C=-D=\frac{i\sqrt{2\tau}}{\sqrt{(1-q)+\sqrt{(1-q)^2+4\,q\,\sigma\,\tau}}}.$$

Since A,B,C,D are purely imaginary, we only need to verify condition $BC<1$ which reads

$$q+2\sqrt{\sigma\tau}-1<\sqrt{(1-q)^2+4q\sigma\tau}. \tag{1.30}$$

This is trivially true when $q+2\sqrt{\sigma\tau}-1<0$. If $q+2\sqrt{\sigma\tau}-1\geq 0$, squaring both sides we get $4(1-q)\sqrt{\sigma\tau}>4(1-q)\sigma\tau$, which holds true as $q>1$ and $0<\sigma\tau<1$.

Thus quadratic harness (X_t) exists by Theorem 1.1, and it remains to verify that its parameters are as claimed. A straightforward calculation shows that (1.7) and (1.8) give the correct values of parameters. It remains to verify that formula (1.9) indeed gives the correct value of parameter γ. Since this calculation is lengthy, we indicate major steps: we write (1.9) as $\gamma-1=(q-1)(1+ABCD)/(1-qABCD)$, and evaluate the right hand side. Substituting values of A,B,C,D we get

$$(q-1)(1+ABCD)/(1-qABCD)=\frac{(1-q)^2+(1+q)\sqrt{(1-q)^2+4q\sigma\tau}}{2q}.$$

Then we use formulas (1.28) and (1.29) to replace $1-q$ and $1+q$ and note that since $\gamma<1-2\sqrt{\sigma\tau}$ we have $\gamma<1-2\sigma\tau$ and

$$\sqrt{(1-q)^2+4q\sigma\tau}=\frac{2(1-\gamma-2\sigma\tau)}{\gamma+\sqrt{(1-\gamma)^2-4\sigma\tau+1}}.$$

This eventually simplifies $(q-1)(1+ABCD)/(1-qABCD)$ to $\gamma-1$, as claimed. $\square$

2. The case of densities

In this section we give an explicit and elementary construction of a quadratic harness on a possibly restricted time interval and under additional restrictions on parameters A,B,C,D.

Proposition 2.1. *Fix parameters $-1<q<1$ and A,B,C,D that are either real or (A,B) or (C,D) are complex conjugate pairs. Without loss of generality, we assume that $|A|\leq|B|$, $|C|\leq|D|$; additionally, we assume that $|BD|<1$. Then the interval*

$$J=\left[\frac{|D|^2-CD}{1-AB|D|^2},\frac{1-CD|B|^2}{|B|^2-AB}\right]$$

has positive length and there exists a unique bounded Markov process $(X_t)_{t\in J}$ with absolutely continuous finite dimensional distributions which satisfies the conclusion of Theorem 1.1.

Remark 2.1. Proposition 2.1 is a special case of Theorem 1.1; the latter may allow to extend the processes constructed here to a wider time-interval:

$$T_0 \le \frac{|D|^2 - CD}{1 - AB|D|^2} \text{ and } \frac{1 - CD|B|^2}{|B|^2 - AB} \le T_1.$$

The easiest way to see the inequalities is to compare the end points of the intervals (2.9) and (1.19) after the Möbius transformation (2.26); for example, $|D|^2 \ge |CD| \ge \max\{0, CD, qCD\}$, where the last term plays a role only when $q < 0$ and $CD < 0$.

The rest of this section contains the construction, ending with the proof of Proposition 2.1.

2.1. **Askey-Wilson densities.**

For complex a and $|q| < 1$ we define

$$\begin{aligned} (a)_n = (a;q)_n &= \begin{cases} \prod_{j=0}^{n-1} (1 - aq^j) & n = 1, 2, \ldots \\ 1 & n = 0 \end{cases} \\ (a)_\infty = (a;q)_\infty &= \prod_{j=0}^{\infty} (1 - aq^j) \end{aligned}$$

and we denote

$$(a_1, a_2, \ldots, a_l)_\infty = (a_1, a_2, \ldots, a_l; q)_\infty = (a_1;q)_\infty (a_2;q)_\infty \ldots (a_l;q)_\infty \ ,$$

$$(a_1, a_2, \ldots, a_l)_n = (a_1, a_2, \ldots, a_l; q)_n = (a_1;q)_\infty (a_2;q)_\infty \ldots (a_l;q)_n \ .$$

The advantage of this notation over the standard product notation lies both in its conciseness and in mnemotechnic simplification rules:

$$\frac{(a,b)_n}{(a,c)_n} = \frac{(b)_n}{(c)_n} \ ,$$

$$(\alpha)_{M+L} = (q^M \alpha)_L \ (\alpha)_M \ , \tag{2.1}$$

and

$$(\alpha)_M = (-\alpha)^M q^{\frac{M(M-1)}{2}} \left(\frac{q}{q^M \alpha}\right)_M \ . \tag{2.2}$$

which often help with calculations. For a reader who is as uncomfortable with this notation as we were at the beginning of this project, we suggest to re-write the formulas for the case $q = 0$. For example, $(a;0)_n$ is either 1 or $(1 - a)$ as $n = 0$ or $n > 0$ respectively. The construction of Markov process for $q = 0$ in itself is quite interesting as the resulting laws are related to the laws that arise in Voiculescu's free probability; the formulas simplify enough so that the integrals can be computed by elementary means, for example by residua.

From Askey and Wilson [4, Theorem 2.1] it follows that if a, b, c, d are complex such that $\max\{|a|, |b|, |c|, |d|\} < 1$ and $-1 < q < 1$, then with $\theta = \theta_x$ such that $\cos\theta = x$

$$\begin{aligned} \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \frac{\left(e^{2i\theta}, e^{-2i\theta}\right)_\infty}{(ae^{i\theta}, ae^{-i\theta},\ be^{i\theta},\ be^{-i\theta},\ ce^{i\theta},\ ce^{-i\theta},\ de^{i\theta},\ de^{-i\theta})_\infty} dx \\ = \frac{2\pi (abcd)_\infty}{(q,\ ab,\ ac,\ ad,\ bc,\ bd,\ cd)_\infty} \ . \end{aligned} \tag{2.3}$$

When $-1 < q < 1$ and a, b, c, d are either real or complex and form two conjugate pairs and $\max\{|a|, |b|, |c|, |d|\} < 1$, the integrand is real and positive. This allows us to define the Askey-Wilson density

$$f(x; a, b, c, d) = \frac{K(a, b, c, d)}{\sqrt{1-x^2}} \left| \frac{(e^{2i\theta})_\infty}{(ae^{i\theta},\ be^{i\theta},\ ce^{i\theta},\ de^{i\theta})_\infty} \right|^2 I_{(-1,1)}(x)\ , \tag{2.4}$$

where

$$K(a, b, c, d) = \frac{(q,\ ab,\ ac,\ ad,\ bc,\ bd,\ cd)_\infty}{2\pi(abcd)_\infty}\ . \tag{2.5}$$

The first two moments are easily computed.

Proposition 2.2. *Suppose X has distribution $f(x; a, b, c, d)$ with parameters a, b, c, d as above. Then the expectation of X is*

$$\mathbb{E}(X) = \frac{a + b + c + d - abc - abd - acd - bcd}{2(1 - abcd)}\ , \tag{2.6}$$

and the variance of X is

$$\operatorname{Var}(X) = \frac{(1-ab)(1-ac)(1-ad)(1-bc)(1-bd)(1-cd)(1-q)}{4(1-abcd)^2(1-abcdq)}\ . \tag{2.7}$$

Proof. If $a = b = c = d = 0$, $\mathbb{E}(X) = 0$ by symmetry. If one of the parameters, say $a \in \mathbb{C}$ is non-zero, we note that

$$\begin{aligned}\left(ae^{i\theta},\ ae^{-i\theta}\right)_\infty = \left(ae^{i\theta},\ ae^{-i\theta}\right)_1 \left(aqe^{i\theta}, aqe^{-i\theta}\right)_\infty \\ = (1 + a^2 - 2ax)\ \left(aqe^{i\theta}, aqe^{-i\theta}\right)_\infty\ .\end{aligned}$$

Therefore by (2.3)

$$\mathbb{E}(1 + a^2 - 2aX) = \frac{K(a, b, c, d)}{K(qa, b, c, d)} = \frac{(1-ab)(1-ac)(1-ad)}{1-abcd}\ .$$

Now (2.6) follows by a simple algebra.

Similarly, for non-zero $a, b \in \mathbb{C}$,

$$\begin{aligned}4ab\operatorname{Var}(X) &= \mathbb{E}\left[(1 + a^2 - 2aX)(1 + b^2 - 2bX)\right] - \mathbb{E}(1+a^2-2aX)\mathbb{E}(1+b^2-2bX) \\ &= \frac{K(a, b, c, d)}{K(qa, qb, c, d)} - \frac{K^2(a, b, c, d)}{K(qa, b, c, d)K(a, qb, c, d)} \\ &= \frac{(1-ab)(1-qab)(1-ac)(1-ad)(1-bc)(1-bd)}{(1-abcd)(1-qabcd)} \\ &\qquad - \frac{(1-ab)^2(1-ac)(1-ad)(1-bc)(1-bd)}{(1-abcd)^2}\ .\end{aligned}$$

Again after simple transformations we arrive at (2.7).

If only one parameter is non-zero but $q \neq 0$, the calculations are similar, starting with $\mathbb{E}\left((1 + a^2 - 2aX)(1 + a^2q^2 - 2aqX)\right)$; when $q = 0$ the density is a re-parametrization of Marchenko-Pastur law [15, (3.3.2)]; we omit the details. If a, b, c, d are zero, $f(x; 0, 0, 0, 0)$ is the orthogonality measure of the continuous q-Hermite polynomials [17, (3.26.3)]; since $H_2(x) = 2xH_1(x) - (1-q)H_0 = 4x^2 - (1-q)$, the second moment is $(1-q)/4$. □

We need a technical result on Askey-Wilson densities inspired by [20, formula (2.4)].

Proposition 2.3. *Let a, b, c, d, q be as above with the additional assumption that the only conjugate pairs possible are $a = \bar{b}$ and/or $c = \bar{d}$ and m is real such that $|m| < 1$. Then with $x = \cos\theta_x$,*

$$\int_{-1}^{1} f\left(x; am, bm, c, d\right) f\left(y; a, b, me^{i\theta_x}, me^{-i\theta_x}\right)\, dx \;=\; f\left(y; a, b, cm, dm\right). \tag{2.8}$$

Proof. We compute the left-hand side of (2.8) expanding the constants $K(am, bm, c, d)$ and $K(a, b, me^{i\theta_x}, me^{-i\theta_x})$ to better show how some factors cancel out. To avoid case-by-case reasoning when complex conjugate pairs are present, we also expand parts of the density without the use of modulus as in (2.3).

The integrand on the left hand side of (2.8) is

$$\frac{(q,\ abm^2,\ acm,\ adm,\ bcm,\ bdm,\ cd)_\infty |\left(e^{2i\theta_x}\right)_\infty|^2}{2\pi(abcdm^2)_\infty\left(ame^{i\theta_x},\ ame^{-i\theta_x},\ bme^{i\theta_x},\ bme^{-i\theta_x},\right)_\infty|\left(ce^{i\theta_x},\ de^{i\theta_x}\right)_\infty|^2}$$

$$\times\frac{\left(q,\ ab,\ ame^{i\theta_x},\ ame^{-i\theta_x},\ bme^{i\theta_x},\ bme^{-i\theta_x},\ m^2\right)_\infty}{2\pi(abm^2)_\infty\sqrt{1-y^2}}$$

$$\times\frac{|\left(e^{2i\theta_y}\right)_\infty|^2}{|\left(ae^{i\theta_y},\ be^{i\theta_y},\ me^{i(\theta_x+\theta_y)},\ me^{i(-\theta_x+\theta_y)}\right)_\infty|^2\sqrt{1-x^2}}.$$

Rearranging the terms we rewrite the left hand side of (2.8) as

$$\frac{(q,\ abm^2,\ acm,\ adm,\ bcm,\ bdm,\ cd,\ q,\ ab,\ m^2)_\infty}{(2\pi)^2(abm^2,\ abcdm^2)_\infty\sqrt{1-y^2}}\times\frac{|\left(e^{2i\theta_y}\right)_\infty|^2}{|\left(ae^{i\theta_y},\ be^{i\theta_y}\right)_\infty|^2}$$
$$\times\int_{-1}^{1}\frac{|\left(e^{2i\theta_x}\right)_\infty|^2}{|\left(me^{i\theta_y}e^{i\theta_x},\ me^{-i\theta_y}e^{i\theta_x},\ ce^{i\theta_x},\ de^{i\theta_x}\right)_\infty|^2}\frac{dx}{\sqrt{1-x^2}}.$$

Now we apply formula (2.3) to this integral, so the left hand side of (2.8) becomes

$$\frac{(q,\ abm^2,\ acm,\ adm,\ bcm,\ bdm,\ cd,\ q,\ ab,\ m^2)_\infty}{(2\pi)^2(abm^2,\ abcdm^2)_\infty\sqrt{1-y^2}}\times\frac{|\left(e^{2i\theta_y}\right)_\infty|^2}{|\left(ae^{i\theta_y},\ be^{i\theta_y}\right)_\infty|^2}$$
$$\times\frac{2\pi(cdm^2)_\infty}{(q,\ m^2,\ mce^{i\theta_y},\ mde^{i\theta_y},\ mce^{-i\theta_y},\ mde^{-i\theta_y},\ cd)_\infty}$$

$$=\frac{(q,\ ab,\ acm,\ adm,\ bcm,\ bdm,\ cdm^2)_\infty}{2\pi(abcdm^2)_\infty\sqrt{1-y^2}}\times\frac{\left(e^{2i\theta_y}\right)_\infty|^2}{|\left(ae^{i\theta_y},\ be^{i\theta_y},\ mce^{i\theta_y},\ mde^{i\theta_y}\right)_\infty|^2},$$

which completes the proof. □

2.2. **Markov processes with Askey-Wilson densities.** We now fix A, B, C, D as in Proposition 2.1. The interval

$$I(A, B, C, D) = \left(|D|^2, \frac{1}{|B|^2}\right) \tag{2.9}$$

is non-empty (here $1/0 = \infty$). For any $t \in I(A, B, C, D)$ and $y \in [-1, 1]$ let

$$p(t, y) = f\left(y; A\sqrt{t}, B\sqrt{t}, \frac{C}{\sqrt{t}}, \frac{D}{\sqrt{t}}\right) \tag{2.10}$$

and for any $s<t$ in $I(A,B,C,D)$ and $x,y\in[-1,1]$ let

$$p(s,x;t,y)=f\left(y;A\sqrt{t},B\sqrt{t},\frac{\sqrt{s}}{\sqrt{t}}e^{i\theta_x},\frac{\sqrt{s}}{\sqrt{t}}e^{-i\theta_x}\right)\,,\quad x=\cos\,\theta_x\,. \tag{2.11}$$

Proposition 2.4. *The family of probability densities $(p(s,x;t,y),\ p(t,y))$ defines a Markov process $(Y_t)_{t\in I}$ on the state space $[-1,1]$. That is, for any $s<t$ from $I(A,B,C,D)$ and $y\in[-1,1]$,*

$$p(t,y)=\int_{-1}^{1}p(s,x;t,y)p(s,x)\,dx \tag{2.12}$$

and for any $s<t<u$ from $I(A,B,C,D)$ and $x,z\in[-1,1]$,

$$p(s,x;u,z)=\int_{-1}^{1}p(t,y;u,z)p(s,x;t,y)\,dy\,. \tag{2.13}$$

Proof. To show (2.12) it suffices just to use the identity (2.8) with $a=A\sqrt{t}$, $b=B\sqrt{t}$, $c=C/\sqrt{s}$, $d=D/\sqrt{s}$, and $m=\sqrt{s/t}\in(0,1)$. We note that this substitution preserves the conjugate pairs in the case of complex parameters, and that by the definition of $I(A,B,C,D)$, parameters $A\sqrt{t}$, $B\sqrt{t}$, $C/\sqrt{s}$ and $D/\sqrt{s}$ have modulus less than one, so (2.8) applies here and gives the desired formula

$$\int_{-1}^{1}f\left(x;A\sqrt{s},B\sqrt{s},\frac{C}{\sqrt{s}},\frac{D}{\sqrt{s}}\right)f\left(y;A\sqrt{t},B\sqrt{t},\frac{\sqrt{s}}{\sqrt{t}}e^{i\theta_x},\frac{\sqrt{s}}{\sqrt{t}}e^{-i\theta_x}\right)\,dx$$
$$=f\left(y;A\sqrt{t},B\sqrt{t},\frac{C}{\sqrt{t}},\frac{D}{\sqrt{t}}\right)\,.$$

To get the second formula (2.13) we again use (2.8) this time with $a=A\sqrt{u}$, $b=\sqrt{u}$, $c=\sqrt{s/t}e^{i\theta_x}$, $d=\sqrt{s/t}e^{-i\theta_x}$, and $m=\sqrt{t/u}$. Thus we arrive at

$$\int_{-1}^{1}f\left(z;A\sqrt{u},B\sqrt{u},\frac{\sqrt{t}}{\sqrt{u}}e^{i\theta_y},\frac{\sqrt{t}}{\sqrt{u}}e^{-i\theta_y}\right)f\left(y;A\sqrt{t},B\sqrt{t},\frac{\sqrt{s}}{\sqrt{t}}e^{i\theta_x},\frac{\sqrt{s}}{\sqrt{t}}e^{-i\theta_x}\right)dy$$
$$=f\left(z;A\sqrt{u},B\sqrt{u},\frac{\sqrt{s}}{\sqrt{u}}e^{i\theta_x},\frac{\sqrt{s}}{\sqrt{u}}e^{-i\theta_s}\right)\,.$$

□

Proposition 2.5. *Let $(Y_t)_{t\in I(A,B,C,D)}$ be the Markov process from Proposition 2.4, with marginal densities (2.10) and transition densities (2.11). For $t\in I(A,B,C,D)$,*

$$\mathbb{E}(Y_t)=\frac{[A+B-AB(C+D)]t+C+D-CD(A+B)}{2\sqrt{t}(1-ABCD)}\,, \tag{2.14}$$

$$\begin{aligned}\operatorname{Var}(Y_t)&\\ &=\frac{(1-q)(1-AC)(1-AD)(1-BC)(1-BD)}{4t(1-ABCD)^2(1-qABCD)}(t-CD)(1-ABt)\end{aligned} \tag{2.15}$$

and for $s,t \in I(A,B,C,D)$, such that $s<t$,

$$\operatorname{Cov}(Y_s,Y_t) = \frac{(1-q)(1-AC)(1-AD)(1-BC)(1-BD)}{4\sqrt{st}(1-ABCD)^2(1-qABCD)}(s-CD)(1-ABt)\ , \tag{2.16}$$

$$\mathbb{E}(Y_t|\mathcal{F}_s) = \frac{(A+B)(t-s)+2(1-ABt)\sqrt{s}Y_s}{2\sqrt{t}(1-ABs)}\ , \tag{2.17}$$

$$\operatorname{Var}(Y_t|\mathcal{F}_s) = \frac{(1-q)(t-s)(1-ABt)}{4t(1-ABs)^2(1-qABs)}(1+A^2s-2A\sqrt{s}Y_s)(1+B^2s-2B\sqrt{s}Y_s)\ . \tag{2.18}$$

Proof. Formulas (2.14) and (2.15) follow, respectively, from (2.6) and (2.7) by taking $a=A\sqrt{t}$, $b=B\sqrt{t}$, $c=C/\sqrt{t}$ and $d=D/\sqrt{t}$.

Similarly, the formulas (2.17) and (2.18) follow, respectively, from (2.6) and (2.7) by taking $a=A\sqrt{t}$, $b=B\sqrt{t}$, $c=\sqrt{\frac{s}{t}}e^{i\theta_x}$ and $d=\sqrt{\frac{s}{t}}e^{-i\theta_x}$.

To obtain the covariance we make use of (2.17) as follows:

$$\operatorname{Cov}(Y_s,Y_t) = \mathbb{E}(Y_s\mathbb{E}(Y_t|\mathcal{F}_s)) - \mathbb{E}(Y_s)\mathbb{E}(Y_t)$$

$$= \left(\frac{(A+B)(t-s)}{2\sqrt{t}(1-ABs)} - \mathbb{E}Y_t\right)\mathbb{E}Y_s + \frac{(1-ABs)\sqrt{s}}{(1-ABt)\sqrt{t}}\left(\operatorname{Var}(Y_s)+[\mathbb{E}Y_s]^2\right)\ .$$

Now the formula (2.16) follows, after a little algebra, from (2.14) and (2.15). ∎

Next we show that the conditional distributions of Y_t given the past and the future of the process is given by an Askey-Wilson density that does not depend on parameters A,B,C,D.

Proposition 2.6. *Let $(Y_t)_{t\in I(A,B,C,D)}$ be the Markov process with marginal densities (2.10) and transition densities (2.11). Then for any $s,t,u \in I(A,B,C,D)$ such that $s<t<u$ the conditional distribution of Y_t given $\mathcal{F}_{s,u}$ has the Askey-Wilson density*

$$f\left(y;\frac{\sqrt{t}}{\sqrt{u}}\exp(i\theta_z),\frac{\sqrt{t}}{\sqrt{u}}\exp(-i\theta_z),\frac{\sqrt{s}}{\sqrt{t}}\exp(i\theta_x),\frac{\sqrt{s}}{\sqrt{t}}\exp(-i\theta_x)\right)\ . \tag{2.19}$$

(Here, $x=\cos\theta_x=Y_s$, $z=\cos\theta_z=Y_u$.) The first two conditional moments have the form

$$\mathbb{E}(Y_t|\mathcal{F}_{s,u}) = \frac{(u-t)\sqrt{s}Y_s+(t-s)\sqrt{u}Y_u}{\sqrt{t}(u-s)}\ , \tag{2.20}$$

$$\operatorname{Var}(Y_t|\mathcal{F}_{s,u}) = \frac{(1-q)(u-t)(t-s)}{t(u-qs)}\left(\frac{1}{4}-\frac{(u\sqrt{s}Y_s-s\sqrt{u}Y_u)(\sqrt{u}Y_u-\sqrt{s}Y_s)}{(u-s)^2}\right)\ . \tag{2.21}$$

Proof. By the Markov property it follows that the conditional density is

$$\frac{p(t,y;u,z)\,p(s,x;t,y)}{p(s,x;u,z)} = \frac{f(z;A\sqrt{u},B\sqrt{u},\frac{\sqrt{t}}{\sqrt{u}}\,e^{i\theta_y},\frac{\sqrt{t}}{\sqrt{u}}\,e^{-i\theta_y})f(y;A\sqrt{t},B\sqrt{t},\frac{\sqrt{s}}{\sqrt{t}}\,e^{i\theta_x},\frac{\sqrt{s}}{\sqrt{t}}\,e^{-i\theta_x})}{f(z;A\sqrt{u},B\sqrt{u},\frac{\sqrt{s}}{\sqrt{u}}\,e^{i\theta_x},\frac{\sqrt{s}}{\sqrt{u}}\,e^{-i\theta_x})}\,.$$

Now the result follows by plugging in the formula above the definition of the Askey-Wilson density (2.4) with suitably chosen parameters. The mean and variance are calculated from (2.6) and (2.7). ☐

Proof of Proposition 2.1. If we define a new process $(Z_t)_{t\in I(A,B,C,D)}$ through

$$Z_t = \frac{2\sqrt{t}}{\sqrt{1-q}}Y_t\,, \tag{2.22}$$

then (Z_t) is Markov and satisfies

$$\mathbb{E}(Z_t|\mathcal{F}_s) = \frac{(A+B)(t-s)}{\sqrt{1-q}(1-ABs)} + \frac{1-ABt}{1-ABs}Z_s$$

so that

$$\left(\frac{AB\sqrt{1-q}Z_t-(A+B)}{1-ABt},\,\mathcal{F}_t\right)$$

is a martingale. Moreover,

$$\operatorname{Var}(Z_t|\mathcal{F}_s) = \frac{(t-s)(1-ABt)}{(1-ABs)^2(1-qABq)}(1+A^2s-A\sqrt{1-q}Z_s)(1+B^2s-B\sqrt{1-q}Z_s)\,.$$

For the double conditioning with respect to the past and future jointly it follows that (Z_t) satisfies quadratic harness conditions

$$\mathbb{E}(Z_t|\mathcal{F}_{s,u}) = \frac{u-t}{u-s}Z_s + \frac{t-s}{u-s}Z_u \tag{2.23}$$

and

$$\operatorname{Var}(Z_t|\mathcal{F}_{s,u}) = \frac{(u-t)(t-s)}{u-qs}\left(1-(1-q)\frac{(uZ_s-sZ_u)(Z_u-Z_s)}{(u-s)^2}\right)\,, \tag{2.24}$$

which correspond to the q-Brownian motion when the process is defined on $(0,\infty)$ and has covariance $\min\{s,t\}$, see [10, Theorem 4.1]. Here, (Z_t) is defined only a finite time domain $I(A,B,C,D)$ and the covariance is different:

$$\operatorname{Cov}(Z_s,Z_t) = \frac{(1-AC)(1-AD)(1-BC)(1-BD)}{(1-ABCD)^2(1-qABCD)}(s-CD)(1-ABt). \tag{2.25}$$

(The law of Z_t will differ from the q-Gaussian law if $|A|+|B|+|C|+|D|>0$.)

The covariance is adjusted by a suitable deterministic time change. Consider a Möbius transformation

$$h(x) = \frac{x-CD}{1-ABx}, \tag{2.26}$$

which for $ABCD < 1$ is an increasing function with the inverse $T(t) = (t+CD)/(1+ABt)$. Note that $J = J(A,B,C,D) = h(I(A,B,C,D))$. For $t \in J(A,B,C,D)$, define

$$
\begin{aligned}
X_t &:= X_{t;A,B,C,D,q} \\
&= \frac{Z_{T(t)} - \mathbb{E}(Z_{T(t)})}{1 - T(t)AB} \times \frac{(1-ABCD)\sqrt{1-qABCD}}{\sqrt{(1-AC)(1-BC)(1-AD)(1-BD)}} \\
&= \frac{\sqrt{1-q}(1+ABt)Z_{T(t)} - (A+B)t - (C+D)}{\sqrt{(1-q)(1-AC)(1-AD)(1-BC)(1-BD)}}\sqrt{1-qABCD}\,.
\end{aligned}
\tag{2.27}
$$

A calculation shows that $(X_t)_{t\in J}$ has unconditional and conditional moments as claimed: formula (1.1) is a consequence of (2.25), and (1.2) follows from (2.23). A much longer calculation shows that (2.24) translates into (1.3) with parameters (1.5-1.9). $\square$

3. Proof of Theorem 1.1

Next, we tackle the issue of extending the quadratic harness from Proposition 2.1 to a larger time interval. The main technical difficulty is that such processes may have a discrete component in their distributions. The construction is based on the Askey-Wilson distribution [4, (2.9)], with slight correction as in [29, (2.5)].

The basic plan of proof of Theorem 1.1 is the same as that of the proof of Proposition 2.1: we define auxiliary Markov process $(Y_t)_{t\in I}$ through a family of Askey-Wilson distributions that satisfy the Chapman-Kolmogorov equations. Then we use formulas (2.22) and (2.27) to define (X_t). The main difference is that due to an overwhelming number of cases that arise with mixed type distributions, we use orthogonal polynomials to deduce all properties we need. (Similar approach was used in [9].)

3.1. **The Askey-Wilson law.**

The Askey-Wilson distribution $\nu(dx;a,b,c,d)$ is the orthogonality measure of the Askey-Wilson polynomials $\{\bar{w}_n(x;a,b,c,d)\}$. Since $\{A_n\}$ and $\{C_n\}$ are bounded sequences, $\nu(dx;a,b,c,d)$ is unique, and compactly supported [16, Theorem 2.5.4]. If $|a|,|b|,|c|,|d|<1$, this is an absolutely continuous measure with density (2.4). In general, $\nu(dx;a,b,c,d)$ may have a discrete component, or be purely discrete as in (4.1). We need also to point out that although parameter a is distinguished in the definition of $\bar{w}_n$, the orthogonality measure is symmetric in all parameters. This can be seen from representation

$$
\bar{w}_n(x;a,b,c,d) = {}_4\varphi_3\left(\begin{matrix} q^{-n}, abcdq^{n-1}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{matrix}; q\right)
\tag{3.1}
$$

and the ${}_4\varphi_3$-Sears transformation [26] which shows how to rescale the Askey-Wilson polynomials to be independent of the order of parameters a,b,c,d without affecting the orthogonality measure.

In general, it is quite difficult to give explicit conditions on a,b,c,d for the existence of $\nu(dx;a,b,c,d)$. By Favard's theorem [16, Theorem 2.5.2], measure $\nu(dx;a,b,c,d)$ exists for all a,b,c,d such that $A_k, C_k \in \mathbb{R}$ for all k and $\prod_{k=0}^{n} A_k C_{k+1} \geq 0$ for all n. Furthermore, it is known that if N is the first integer such that $A_{N-1}C_N = 0$, then $\nu(dx;a,b,c,d;q)$ is a discrete measure supported on the finite set of N zeros of the polynomial $\bar{w}_N(x)$.

Under (1.14), if a, b, c, d are either real or come in complex conjugate pairs and $a \neq 0$ then $A_k > 0$ and $C_k \in \mathbb{R}$ for all k; so in this case the Favard condition is

$$\prod_{k=1}^{n} C_k \geq 0 \text{ for all } n \geq 1. \tag{3.2}$$

A simple sufficient condition for (3.2) is that (1.14) holds and in addition

$$bc, qbc, bd, qbd, cd, qcd \in \mathbb{C} \setminus [1, \infty)\,. \tag{3.3}$$

Under this condition, if a, b, c, d are either real or come in complex conjugate pairs, then the corresponding measure of orthogonality $\nu(dx; a, b, c, d)$ exists. Unfortunately, this simple condition is not good enough for our purposes – we need to allow also Askey-Wilson laws with finite support as in [3]. In fact, such laws describe transitions of the Markov chain in the atomic part.

We now state conditions that cover all the cases needed in this paper. Let $m_1 = m_1(a, b, c, d)$ denote the number of the products ab, ac, ad, bc, bd, cd that fall into $[1, \infty)$, and let $m_2 = m_2(a, b, c, d)$ denote the number of the products qab, qac, qad, qbc, qbd, qcd that fall into $[1, \infty)$.

Lemma 3.1. *Assume $abcd < 1$ and $qabcd < 1$. If $q \geq 0$ then either $m_1 = 0$ or $m_1 = 2$; if $q \leq 0$ and $m_1 m_2 = 0$ then there are only three possible choices: $(m_1, m_2) = (0, 0), (0, 2), (2, 0)$. Only the following cases are possible:*

(i) *If $q \geq 0$ and m_1 is zero, then $\nu(dx; a, b, c, d)$ exists.*
(ii) *If $q < 0$ and $m_1 = m_2 = 0$, then $\nu(dx; a, b, c, d)$ exists.*
(iii) *If $q \geq 0$ and $m_1 = 2$, then $a, b, c, d \in \mathbb{R}$. In this case, $\nu(dx; a, b, c, d)$ is well defined if either $q = 0$ or the smaller of the two products that fall into $[1, \infty)$ is of the form $1/q^N$ and in this latter case $\nu(dx; a, b, c, d)$ is a purely discrete measure with $N + 1$ atoms.*
(iv) *If $q < 0$ and $m_1 m_2 > 0$ then $\nu(dx; a, b, c, d)$ fails to exists.*
(v) *If $q < 0$ and $m_1 = 2, m_2 = 0$ then $a, b, c, d \in \mathbb{R}$. In this case, $\nu(dx; a, b, c, d)$ is well defined if the smaller of the two products in $[1, \infty)$, say ab, is $1/q^N$, with even N. Then $\nu(dx; a, b, c, d)$ is a purely discrete measure with $N + 1$ atoms.*
(vi) *If $q < 0$ $m_1 = 0, m_2 = 2$, then $a, b, c, d \in \mathbb{R}$. In this case, $\nu(dx; a, b, c, d)$ is well defined if the smaller of the two products in $[1, \infty)$, say qab, is $1/q^N$, with even N. Then $\nu(dx; a, b, c, d)$ is a purely discrete measure with $N + 2$ atoms.*

Proof. The fact that $m_1 = 0$ or $m_1 = 2$ is an elementary observation. Existence of $\nu(dx; a, b, c, d)$ follows by inspection of $\prod_{k=1}^{n} A_{k-1} C_k$. If $m_2 > 0$ then in cases (iii) and (v) $\prod_{k=1}^{n} A_{k-1} C_k > 0$ for $n < N$ and is zero for $n \geq N + 1$. In case (vi), $\prod_{k=1}^{n} A_{k-1} C_k = 0$ for all $n \geq N + 2$.

□

According to Askey and Wilson [4] the orthogonality law is

$$\nu(dx; a, b, c, d) = f(x; a, b, c, d) 1_{|x| \leq 1} + \sum_{x \in F(a,b,c,d)} p(x) \delta_x;$$

here $F = F(a, b, c, d)$ is a finite or empty set, and the density f is given by (2.4). The continuous component is absent when $K(a, b, c, d) = 0$ (recall (2.5)). The atoms

occur for parameters which are real and outside of $[-1,1]$. Each such parameter, say a, leads to a finite set of atoms at

$$x_j = \frac{aq^j + (aq^j)^{-1}}{2}, \tag{3.4}$$

with $j \geq 0$ such that $|q^j a| \geq 1$ and the corresponding probabilities are

$$p(x_0) = \frac{(a^{-2}, bc, bd, cd)_\infty}{(b/a, c/a, d/a, abcd)_\infty}, \tag{3.5}$$

$$p(x_j) = p(x_0)\frac{(a^2, ab, ac, ad)_j (1-a^2q^{2j})}{(q, qa/b, qa/c, qa/d)_j (1-a^2)} \left(\frac{q}{abcd}\right)^j, \ j \geq 0. \tag{3.6}$$

(This formula needs to be re-written in an equivalent form to cover the cases when $abcd = 0$. It is convenient to count as "atom" the case $|q^j a| = 1$ even though the corresponding probability is 0.)

We remark that if X has distribution $\nu(dx; a, b, c, d)$ then formulas for $\mathbb{E}(X)$ and $\mathrm{Var}(X)$ from Proposition 2.2 hold now for all admissible choices of parameters a, b, c, d, as these expressions can equivalently be derived from the fact that the first two Askey-Wilson polynomials integrate to zero.

3.2. **Construction of Markov process.** Recall $I = I(A, B, C, D; q)$ from (1.19). As in Section 2, we first construct the auxiliary Markov process $(Y_t)_{t\in I}$. We request that the univariate law π_t of Y_t is the Askey-Wilson law

$$\pi_t(dy) = \nu\left(dy; A\sqrt{t}, B\sqrt{t}, \frac{C}{\sqrt{t}}, \frac{D}{\sqrt{t}}\right). \tag{3.7}$$

In order to ensure that this univariate law exists, we use condition (3.3). This condition is fulfilled when (1.4) holds and the admissible range of values of t is the interval I from (1.19). (The endpoints (1.12) and (1.13) were computed by applying Möbius transformation (2.26) to the endpoints of I.)

For $t \in I$, let U_t be the support of $\pi_t(dy)$. Under the assumption (1.4), this set can be described quite explicitly using the already mentioned results of Askey-Wilson [4]: U_t is the union of $[-1,1]$ and a finite or empty set F_t of points that are of the form

$$x_j(t) = \frac{1}{2}\left(B\sqrt{t}q^j + \frac{1}{B\sqrt{t}q^j}\right) \text{ or } u_j(t) = \frac{1}{2}\left(\frac{Dq^j}{\sqrt{t}} + \frac{\sqrt{t}}{Dq^j}\right), \tag{3.8}$$

$$y_j(t) = \frac{1}{2}\left(A\sqrt{t}q^j + \frac{1}{A\sqrt{t}q^j}\right) \text{ or } v_j(t) = \frac{1}{2}\left(\frac{Cq^j}{\sqrt{t}} + \frac{\sqrt{t}}{Cq^j}\right). \tag{3.9}$$

There is at most finite number of points of each type. However, not all such atoms can occur simultaneously. All possibilities are listed in the following lemma.

Lemma 3.2. *Under the assumptions of Theorem 1.1, without loss of generality assume $|A| \leq |B|$ and $|C| \leq |D|$. Then the following atoms occur:*

- *Atoms $u_j(t)$ appear for $D, C \in \mathbb{R}$, and $t \in I$ that satisfy $t < D^2$; allowed $j \geq 0$ satisfy $D^2q^{2j} > t$.*
- *Atoms $v_j(t)$ appear for $D, C \in \mathbb{R}$, and $t \in I$ that satisfy $t < C^2$; allowed $j \geq 0$ satisfy $C^2q^{2j} > t$.*
- *Atoms $x_j(t)$ appear for $A, B \in \mathbb{R}$, and $t \in I$ that satisfy $t > 1/B^2$; allowed $j \geq 0$ satisfy $tB^2q^{2j} > 1$.*

- *Atoms $y_j(t)$ appear for $A, B \in \mathbb{R}$, and $t \in I$ that satisfy $t > 1/A^2$; allowed $j \geq 0$ satisfy $tA^2q^{2j} > 1$.*

(The actual number of cases is much larger as in proofs one needs to consider all nine possible choices for the end points of the time interval I.)

Next, we specify the transition probabilities of Y_t.

Proposition 3.3. *For $s < t$, $s, t \in I$ and any real $x \in U_s$ measures*
$$P_{s,t}(x, dy) = \nu\left(dy; A\sqrt{t}, B\sqrt{t}, \sqrt{\frac{s}{t}}(x + \sqrt{x^2-1}), \sqrt{\frac{s}{t}}(x - \sqrt{x^2-1})\right)$$
are well defined.

Proof. Clearly, $P_{s,t}(x, dy)$ for $x \in [-1, 1]$ is absolutely continuous and has density (2.11). This covers all possibilities when (A, B) and (C, D) are conjugate pairs or when $|A|\sqrt{s}, |B|\sqrt{s}, |C|/\sqrt{s}, |D|/\sqrt{s} < 1$, as then $U_s \subset [-1, 1]$.

It remains to consider x in the atomic part of $\pi_s(dx)$. Relabeling the parameters if necessary, we may assume $|A| \leq |B|$ and $|C| \leq |D|$. For each of the cases listed in Lemma 3.2, we need to show that the choice of parameters $a = A\sqrt{t}$, $b = B\sqrt{t}$, $c = \sqrt{\frac{s}{t}}(x + \sqrt{x^2-1})$, $d = \sqrt{\frac{s}{t}}(x - \sqrt{x^2-1})$ leads to non-negative products $\prod_{k=0}^n A_kC_{k+1} \geq 0$ (recall (1.16) and (1.17)). We check this by considering all possible cases for the endpoints of I and all admissible choices of x from the atoms of measure π_s. In the majority of these cases, condition (3.3) holds so in fact $A_kC_{k+1} > 0$ for all k.

Here is one sample case that illustrates what kind of reasoning is involved in the "simpler cases" where (3.3) holds, and one example for more complicated case where (3.3) fails.

- **Case** $CD < 0$, $AB < 0$, $q \geq 0$**:** In this case, A, B, C, D are real, $I = (0, \infty)$ and assumption $ABCD < 1$ implies $D^2 < 1/B^2$. A number of cases arises from Lemma 3.2, and we present only one of them:
  - **Sub-Case** $x = v_j(s)$**:** Then $0 < s < C^2$ and the Askey-Wilson parameters of $P_{s,t}(x; dy)$ are
  $$a = A\sqrt{t},\ b = B\sqrt{t},\ c = \frac{q^jC}{\sqrt{t}},\ d = \frac{s}{Cq^j\sqrt{t}}.$$
  Thus
  $$ab = ABt < 0 < 1,\ ac = ACq^j,\ ad = \frac{As}{Cq^j},\ bc = q^jBC,\ bd = \frac{Bs}{Cq^j},\ cd = s/t < 1$$
  and
  $$qab = qABt < 0 < 1,\ qac = ACq^{j+1} < ACq^j,\ qad = \frac{As}{Cq^{j-1}},\ qbc = q^{j+1}BC,$$
  $$qbd = \frac{Bs}{Cq^{j-1}},\ qcd = qs/t < 1\,.$$
  Since $s < C^2q^{2j}$, this implies $|ad| < |A|\sqrt{s} < 1$ as $s < C^2 \leq D^2 < 1/B^2 \leq 1/A^2$. For the same reason, $|bd| < 1$. Of course, $|qad| < |ad| < 1$, $|qbd| < |bd| < 1$
  Finally, since $AC, qAC, BC, qBC < 1$, we have $ac, qac, bc, qbc < 1$. Thus by Lemma 3.1(i), $P_{s,t}(x, dy)$ is well defined.

  (We omit other elementary but lengthy sub-cases that lead to (3.3).)

- **Case** $A, B, C, D \in \mathbb{R}$**,** $AB > 0$**,** $CD > 0$**:** Here $I(A, B, C, D) = (CD, 1/(AB))$ is non-empty. Again a number of cases arises from Lemma 3.2, of which we present only one.
  - **Sub-Case** $x = x_j(s)$**:** Here $B^2 < s < t < 1/(AB)$ and $B\sqrt{s}|q^j| \geq 1$. Then the Askey-Wilson parameters of $P_{s,t}(x; dy)$ are

$$a = A\sqrt{t},\ b = B\sqrt{t},\ c = \frac{sq^j B}{\sqrt{t}},\ d = \frac{1}{q^j B\sqrt{t}}.$$

Here, Lemma 3.1(ii) or Lemma 3.1(v) applies with $m_1 = 2$ and $m_2 = 0$ when $q \geq 0$ or j is even, and Lemma 3.1(vi) applies with $m_1 = 0$, $m_2 = 2$ when $q < 0$ and j is odd. To see this, we look at the two lists of pairwise products:

$$ab = ABt < 1,\ ac = ABsq^j < 1,\ ad = \frac{A}{Bq^j},\ bc = sq^j B^2,\ bd = 1/q^j,$$
$$cd = s/t < 1$$

and

$$qab = qABt < 0 < 1,\ qac = ABsq^{j+1} < 1,\ qad = \frac{A}{Bq^{j-1}},\ qbc = sq^{j+1}B^2,$$
$$qbd = 1/q^{j-1},\ qcd = qs/t < 1\,.$$

Since $|A| \leq |B|$ we see that $1/(AB) < 1/A^2$ so $|A| < 1/\sqrt{s}$ and $|\frac{A}{Bq^j}| < \frac{\sqrt{s}}{|q^j B|} \leq 1$. This shows that $ad < 1$ and $qad < 1$.
It is clear that both $bc, bd > 1$ when $q > 0$ or j is even, and that $qbc, qbd > 1$ when $q < 0$ and j is odd. Thus by Lemma 3.1, $P_{s,t}(x, dy)$ is well defined.

Other cases are handled similarly and are omitted. □

In the continuous case, $p_t(dy) = p(t, y)dy$ and $P_{s,t}(x, dy) = p(s, x; t, y)dy$ correspond to (2.10) and (2.11) respectively. We now extend Proposition 2.4 to a larger set of measures.

Proposition 3.4. *The family of probability measures $\pi_t(dy)$ together with the family of transition probabilities $P_{s,t}(x, dy)$ defines a Markov process $(Y_t)_{t\in I}$ on $\bigcup_{t\in I} U_t$. That is, $P_{s,t}(x, dy)$ is defined for all $x \in U_s$. For any $s < t$ from $I(A, B, C, D)$ and Borel set $V \subset \mathbb{R}$,*

$$\pi_t(V) = \int_{\mathbb{R}} P_{s,t}(x, V)\pi_s(dx) \tag{3.10}$$

and for $s < t < u$ from I,

$$P_{s,u}(x, V) = \int_{U_t} P_{t,u}(y, V)P_{s,t}(x, dy) \text{ for all } x \in U_s\ . \tag{3.11}$$

To prove this result we follow the same plan that we used in [9] and we will use similar notation. We deduce all necessary information from the orthogonal polynomials. Consider two families of polynomials:

$$p_n(x; t) = \bar{w}_n\left(x; A\sqrt{t}, B\sqrt{t}, \frac{C}{\sqrt{t}}, \frac{D}{\sqrt{t}}\right)$$

with $\bar{w}_n$ defined by (1.15), and Q_n that are essentially

$$\bar{w}_n\left(y; A\sqrt{t}, B\sqrt{t}, \sqrt{\frac{s}{t}}(x+\sqrt{x^2-1}), \sqrt{\frac{s}{t}}(x-\sqrt{x^2-1})\right)$$

multiplied by

$$\left(\sqrt{\frac{s}{t}}(x+\sqrt{x^2-1}), \sqrt{\frac{s}{t}}(x-\sqrt{x^2-1})\right)_n .$$

To avoid singularities that may arise when x is in the discrete part of U_s, formally we define Q_n for all x by $Q_{-1}=0$, $Q_0=1$ and by the three step recurrence:

$$\begin{aligned} &2yQ_n(y;x,t,s) \\ &\quad = \bar{A}_n Q_{n+1}(y;x,t,s) + (a+1/a-A_n-C_n)Q_n(y;x,t,s) + \bar{C}_n Q_{n-1}(y;x,t,s) \end{aligned} \tag{3.12}$$

with A_n, C_n defined by (1.16) and (1.17),

$$\bar{A}_n = \frac{(1-abcdq^{n-1})(1-abq^n)}{a(1-abcdq^{2n-1})(1-abcdq^{2n})}, \tag{3.13}$$

$$\bar{C}_n = \frac{a(1-q^n)(1-acq^{n-1})(1-adq^{n-1})(1-bcq^{n-1})(1-bdq^{n-1})(1-cdq^{n-1})}{(1-abcdq^{2n-2})(1-abcdq^{2n-1})}, \tag{3.14}$$

and with $a=A\sqrt{t}$, $b=B\sqrt{t}$, $c=\sqrt{\frac{s}{t}}(x+\sqrt{x^2-1})$, and $d=\sqrt{\frac{s}{t}}(x-\sqrt{x^2-1})$.

Polynomials $\{p_n\}$ are orthogonal with respect to $\pi_t(dx)$ and polynomials $\{Q_n\}$ are orthogonal with respect to $P_{s,t}(x,dy)$ when the parameters take appropriate values.

The connection coefficients are due to Askey and Wilson [4].

Lemma 3.5. *For $n \geq 1$,*

$$Q_n(y;x,t,s) = \sum_{k=1}^{n} b_{n,k}(x,s)\left(p_k(y;t)-p_k(x;s)\right) , \tag{3.15}$$

where $b_{n,k}(x,s)$ do not depend on t and $b_{n,n}(x,s) \neq 0$ does not depend on x.

Proof. An adaptation of [4, Formula (6.3)] with $\beta = b$ shows that

$$Q_n(y;x,t,s) = \sum_{k=0}^{n} b_{n,k}(x,s)p_k(y;t) \tag{3.16}$$

with the connection coefficients

$$\begin{aligned} b_{n,k}(x,s) = (-1)^k q^{k(k+1)/2}(a\gamma, a\delta)_n &\frac{(q^{-n}, q^{n-1}ab\gamma\delta, ac, ad)_k}{(q, q^{k-1}abcd, a\delta, a\gamma)_k} \\ &\times {}_4\varphi_3\left(\begin{matrix} q^{k-n}, ab\gamma\delta q^{n+k-1}, acq^k, adq^k \\ q^{2k}abcd, a\gamma q^k, a\delta q^k \end{matrix}; q\right), \end{aligned}$$

which do not depend on t, as

$$\begin{aligned} ab\gamma\delta &= ABCD, & abcd &= ABs, \\ a\gamma &= A\sqrt{s}(x+\sqrt{x^2-1}), & ac &= AC, \\ a\delta &= A\sqrt{s}(x-\sqrt{x^2-1}), & ad &= AD. \end{aligned}$$

By (2.2),

$$b_{n,n}(x,s) = (-1)^n q^{n(n+1)/2} \frac{(q^{-n}, q^{n-1}ABCD)_n}{(q, q^{n-1}ABs, AC, AD)_n} = \frac{(q, q^{n-1}ABCD)_n}{(q, q^{n-1}ABs, AC, AD)_n} \neq 0$$

does not depend on x.

We now observe that (3.12) gives $Q_1(x;x,s,s)=0$, as $a+1/a-A_0-C_0 = 2x$ when $t=s$, and $Q_2(x;x,s,s)=0$, as $\bar{C}_1(a,b,c,d)=0$ when $t=s$. So $Q_n(x;x,s,s)=0$ for $n \geq 1$, and (3.16) implies

$$\sum_{k=0}^{n} b_{n,k}(x,s) p_k(x;s) = 0.$$

Subtracting this identity from (3.16) we get (3.15). □

We also need the following generalization of the projection formula [20].

Proposition 3.6. *Suppose that A, B, C, D satisfy the assumptions in Theorem 1.1. For $x \in U_s$,*

$$\int_{\mathbb{R}} p_n(y;t) P_{s,t}(x,dy) = p_n(x;s)\,. \tag{3.17}$$

Proof. Since $x \in U_s$, from Proposition 3.3, measures $P_{s,t}(x,dy)$ are well defined. The formula holds true for $n=0$. Suppose it holds true for some $n \geq 0$. By induction assumption and orthogonality of polynomials $\{Q_n\}$,

$$\begin{aligned} 0 &= \int_{\mathbb{R}} Q_{n+1}(y;x,t,s) P_{s,t}(x,dy) \\ &= b_{n+1,n+1}(x,s) \int_{\mathbb{R}} \left(p_{n+1}(y;t) - p_{n+1}(x;s)\right) P_{s,t}(x,dy) \\ &= b_{n+1,n+1}(x,s) \left(\int_{\mathbb{R}} p_{n+1}(y;t) P_{s,t}(x,dy) - p_{n+1}(x;s) \right). \end{aligned}$$

□

Proof of Proposition 3.4. This proof follows the scheme of the proof of [9, Proposition 2.5] To prove (3.10), let $\mu(V) = \int_{\mathbb{R}} P_{s,t}(x,V)\pi_s(dx)$ and note that by orthogonality, $\int_{\mathbb{R}} p_n(x;s)\pi_s(dx) = 0$ for all $n \geq 1$. Then from (3.17),

$$\int_{\mathbb{R}} p_n(y;t)\mu(dy) = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} p_n(y,t) P_{s,t}(x,dy) \right) \pi_s(dx) = \int_{\mathbb{R}} p_n(x;s)\pi_s(dx) = 0.$$

Since $\int_{\mathbb{R}} p_n(y;t)\pi_t(dy) = 0$, this shows that all moments of $\mu(dy)$ and $\pi_t(dy)$ are the same. By the uniqueness of the moment problem for compactly supported measures, $\mu(dy) = \pi_t(dy)$, as claimed.

To prove (3.11), we first note that for $x \in U_s$, $P_{s,t}(x,U_t) = 1$; this can be seen by analyzing the locations of atoms, which arise either from the values of $A\sqrt{t}$ or $B\sqrt{t} > 1$ or from x being one of the atoms of U_s. (Alternatively, use (3.10).)

Fix $x \in U_s$ and let $\mu(V) = \int_{U_t} P_{t,u}(y, V) P_{s,t}(x, dy)$. Then, by (3.16) for $n \geq 1$ and (3.17) used twice,

$$\int_{\mathbb{R}} Q_n(z; x, u, s)\mu(dz)$$
$$= \int_{U_t} \int_{\mathbb{R}} \sum_{k=1}^{n} b_{n,k}(x, s) \left(p_k(z; u) - p_k(x; s)\right) P_{t,u}(y, dz) P_{s,t}(x, dy)$$
$$= \int_{U_t} \sum_{k=1}^{n} b_{n,k}(x, s) \left(p_k(y; t) - p_k(x; s)\right) P_{s,t}(x, dy)$$
$$= \int_{\mathbb{R}} \sum_{k=1}^{n} b_{n,k}(x, s) \left(p_k(y; t) - p_k(x; s)\right) P_{s,t}(x, dy) = 0\,.$$

Thus the moments of μ and $P_{s,u}(x, dz)$ are equal which by the method of moments ends the proof.

□

3.3. **Proofs of Theorem 1.1 and Theorem 1.2.**

Proof of Theorem 1.2. If A, B, C, D satisfy the assumptions in Theorem 1.1, by Proposition 3.4, there exists a Markov process (Y_t) with orthogonal polynomials $\{p_n(x; t)\}$. From (3.17) we see that $\{p_n(x; t)\}$ are also martingale polynomials for (Y_t). With Z_t defined by (2.22), polynomials $r_n(x; t)$ inherit the orthogonal martingale property, as $r_n(Z_t; t) = p_n(Y_t; t)$. □

Proof of Theorem 1.1. From Proposition 3.4, we already have the Markov process (Y_t) with Askey-Wilson transition probabilities. Thus the mean and covariance are (2.14) and (2.16). Formulas (2.22) and (2.27) will therefore give us the process $(X_t)_{t \in J}$ with the correct covariance.

It remains to verify asserted properties of conditional moments. Again transformation (2.27) will imply (1.3), provided (Z_t) satisfies (2.24). For the proof of the latter we use orthogonal martingale polynomials (1.20). Our proof is closely related to [8, Theorem 2.3]. We begin by writing the three step recurrence as

$$x r_n(x; t) = (\alpha_n t + \beta_n) r_{n+1}(x; t) + (\gamma_n t + \delta_n) r(x; t) + (\varepsilon_n t + \varphi_n) r_{n-1}(x; t)$$

which amounts to decomposing the Jacobi matrix $\mathbf{J}_t$ of $\{r_n(x; t)\}$ as $t\mathbf{x} + \mathbf{y}$. From (1.15) with $a = A\sqrt{t}$, $b = B\sqrt{t}$, $c = C/\sqrt{t}$ and $d = D/\sqrt{t}$ we read out the coefficients:

$$\begin{aligned}
\alpha_n &= -ABq^n \beta_n\,, \\
\beta_n &= \frac{\left(1 - ABCDq^{n-1}\right)(1 - ACq^n)(1 - ADq^n)}{A\sqrt{1-q}\,(1 - ABCDq^{2n})(1 - ABCDq^{2n-1})}\,, \\
\epsilon_n &= \frac{A\left(1 - BCq^{n-1}\right)\left(1 - BDq^{n-1}\right)(1 - q^n)}{\sqrt{1-q}\,(1 - ABCDq^{2n-2})(1 - ABCDq^{2n-1})}\,, \\
\varphi_n &= -CDq^{n-1}\varepsilon_n\,, \\
\gamma_n &= A/\sqrt{1-q} - \alpha_n - \varepsilon_n\,, \\
\delta_n &= \frac{1}{A\sqrt{1-q}} - \beta_n - \varphi_n\,.
\end{aligned}$$

A calculation verifies the q-commutation equation $[\mathbf{x},\mathbf{y}]_q=\mathbf{I}$ for the two components of the Jacobi matrix. In terms of the coefficients this amounts to

$$\alpha_n\beta_{n-1} = q\alpha_{n-1}\beta_n\,, \tag{3.18}$$
$$\beta_n\gamma_{n+1}+\alpha_n\delta_n = q(\beta_n\gamma_n+\alpha_n\delta_{n+1})\,, \tag{3.19}$$
$$\gamma_n\delta_n+\beta_n\epsilon_{n+1}+\alpha_{n-1}\phi_n = q(\gamma_n\delta_n+\beta_{n-1}\epsilon_n+\alpha_n\phi_{n+1})+1\,, \tag{3.20}$$
$$\delta_n\epsilon_n+\gamma_{n-1}\phi_n = q(\delta_{n-1}\epsilon_n+\gamma_n\phi_n)\,, \tag{3.21}$$
$$\epsilon_n\phi_{n+1} = q\epsilon_{n+1}\phi_n\,. \tag{3.22}$$

For a similar calculation see [30, Section 4.2]. A more general q-commutation equation $[\mathbf{x},\mathbf{y}]_q=\mathbf{I}+\theta\mathbf{x}+\eta\mathbf{y}+\tau\mathbf{x}^2+\sigma\mathbf{y}^2$ appears in [8, (2.22)-(2.26)].

For compactly supported measures, conditional moments can be now read out from the properties of the Jacobi matrices: formula (1.2) follows from $\mathbf{J}_t=t\mathbf{x}+\mathbf{y}$, and formula (1.3) follows from $[\mathbf{x},\mathbf{y}]_q=\mathbf{I}$. This can be seen from the proof of [8, Lemma 3.4] but for reader's convenience we include some details.

Denote by $\mathbf{r}(x;t)=[r_0(x;t),r_1(x;t),\dots]$. Then the three step recurrence is $x\mathbf{r}(x;t)=\mathbf{r}(x;t)\mathbf{J}_t$ and the martingale polynomial property from Theorem 1.2 says that $\mathbb{E}(\mathbf{r}(X_u;u)|\mathcal{F}_t)=\mathbf{r}(X_t;t)$. (Here we take all operations componentwise.)

To verify (1.2) for compactly supported measures it suffices to verify that

$$\mathbb{E}(X_t\mathbf{r}(X_u;u)|\mathcal{F}_s)=\frac{u-t}{u-s}\mathbb{E}(X_s\mathbf{r}(X_u;u)|\mathcal{F}_s)+\frac{t-s}{u-s}\mathbb{E}(X_u\mathbf{r}(X_u;u)|\mathcal{F}_s).$$

Using martingale property and the three step recurrence, this is equivalent to $\mathbf{r}(X_s;s)\mathbf{J}_t=\mathbf{r}(X_s;s)(\frac{u-t}{u-s}\mathbf{J}_s+\frac{t-s}{u-1}\mathbf{J}_u)$ which holds true as $\mathbf{J}_t=\frac{u-t}{u-s}\mathbf{J}_s+\frac{t-s}{u-1}\mathbf{J}_u$ for linear expressions in t.

To verify (2.24) we write it as

$$\mathbb{E}(Z_t^2|\mathcal{F}_{s,u})=\frac{(u-t)(u-qt)Z_s^2}{(u-s)(u-qs)}+\frac{(q+1)(t-s)(u-t)Z_uZ_s}{(u-s)(u-qs)}+\frac{(t-s)(t-qs)Z_u^2}{(u-s)(u-qs)}+\frac{(t-s)(u-t)}{u-qs}. \tag{3.23}$$

For compactly supported laws, it suffices therefore to verify that

$$\begin{aligned}\mathbb{E}(Z_t^2\mathbf{r}(X_u;u)|\mathcal{F}_s)&=\frac{(u-t)(u-qt)}{(u-s)(u-qs)}\mathbb{E}(Z_s^2\mathbf{r}(X_u;u)|\mathcal{F}_s)\\&+\frac{(q+1)(t-s)(u-t)}{(u-s)(u-qs)}\mathbb{E}(Z_uZ_s\mathbf{r}(X_u;u)|\mathcal{F}_s)+\frac{(t-s)(t-qs)}{(u-s)(u-qs)}\\&+\frac{(t-s)(u-t)}{u-qs}\mathbb{E}(Z_u^2\mathbf{r}(X_u;u)|\mathcal{F}_s).\end{aligned} \tag{3.24}$$

Again, we can write this using the Jacobi matrices and martingale property as

$$\begin{aligned}\mathbf{r}(X_s;s)\mathbf{J}_t^2=\mathbf{r}(X_s;s)\Big(&\frac{(u-t)(u-qt)}{(u-s)(u-qs)}\mathbf{J}_s^2+\frac{(q+1)(t-s)(u-t)}{(u-s)(u-qs)}\mathbf{J}_s\mathbf{J}_u\\&+\frac{(t-s)(t-qs)}{(u-s)(u-qs)}\mathbf{J}_u^2\Big)+\frac{(t-s)(u-t)}{u-qs}\mathbf{r}(X_s;s).\end{aligned} \tag{3.25}$$

A calculation shows that for $\mathbf{J}_t = t\mathbf{x}+\mathbf{y}$, the q-commutation equation $[\mathbf{x},\mathbf{y}]_q = \mathbf{I}$ is equivalent to

$$\mathbf{J}_t^2 = \frac{(u-t)(u-qt)}{(u-s)(u-qs)}\mathbf{J}_s^2 + \frac{(q+1)(t-s)(u-t)}{(u-s)(u-qs)}\mathbf{J}_s\mathbf{J}_u + \frac{(t-s)(t-qs)}{(u-s)(u-qs)}\mathbf{J}_u^2 + \frac{(t-s)(u-t)}{u-qs}\mathbf{I}, \tag{3.26}$$

so (3.25) holds.

Uniqueness of (X_t) follows from the fact that by [8, Theorem 4.1] each such process has orthogonal martingale polynomials; from martingale property (3.17) all joint moments are determined uniquely, and correspond to finite dimensional distributions with compactly supported marginals.

□

We remark that the following version of Propositions 2.6 and 4.5 would shorten the proof of Theorem 1.1.

Conjecture 3.1. Let $(Y_t)_{t\in I}$ be the Markov process from Proposition 3.4. Then for any $s<t<u$ from $I(A,B,C,D)$, the conditional distribution of Y_t given $\mathcal{F}_{s,u}$ is

$$\nu\left(y; \frac{z\sqrt{t}}{\sqrt{u}}, \frac{\sqrt{t}}{z\sqrt{u}}, \frac{x\sqrt{s}}{\sqrt{t}}, \frac{\sqrt{s}}{x\sqrt{t}}\right). \tag{3.27}$$

(Here, $x = Y_s + \sqrt{Y_s^2-1}$, $z = Y_u + \sqrt{Y_u^2-1}$.)

4. Purely discrete case

Assumption (1.4) arises from the positivity condition for the Askey-Wilson recurrence for which it is difficult to give general explicit conditions. The following result exhibits additional quadratic harnesses when condition (1.4) is not satisfied.

Proposition 4.1. *Suppose $q,A,B,C,D>0$ and $ABCD<1$. Suppose that there are exactly two numbers among the four products AC,AD,BC,BD that are larger than one, and that the smaller of the two, say, AD is of the form $1/q^N$ for some integer $N\geq 0$. If $Aq^N>1$ then there exists a Markov process $(X_t)_{t\in(0,\infty)}$ with discrete univariate distributions supported on $N+1$ points such that* (1.1) (1.2) *hold, and* (1.3) *holds with parameters $\eta,\theta,\sigma,\tau,\gamma$ given by* (1.5) *through* (1.9).

After re-labeling the parameters, without loss of generality we will assume that $0<A<B$, $0<C<D$, $AC<1$, $BC<1$, $AD=1/q^N$, so that $BD>1/q^N$.

Remark 4.1. Discrete Askey-Wilson distributions $\nu(dx; A\sqrt{t}, B\sqrt{t}, C/\sqrt{t}, D/\sqrt{t})$ may be defined also when $ABCD>1$, a case which we do not consider in this paper.

4.1. Discrete Askey-Wilson distribution.

The discrete Askey-Wilson distribution $\nu_n(dx;a,b,c,d)$ arises in several situations, including the case described in Lemma 3.1(iii). This distribution was studied in detail by Askey and Wilson [3], and summarized in [4].

Here we consider parameters $a,b,c,d>0$ and $0<q<1$ such that $ad=1/q^N$ and

$$q^N a>1, q^N/(bc)>1,\ ac<1,\ q^N a/b>1,\ q^N a/c>1,\ q^N ab>1.$$

Note that this implies $abcd < 1$ and

$$ad = 1/q^N > 1,\ ac < 1,\ bc < 1,\ bd < 1,\ cd < 1,\ ab > 1/q^N,$$

so from Lemma 3.1(iii), Askey-Wilson law $\nu(dx; a, b, c, d) = \nu(dx; a, b, c, 1/(aq^N))$ is well defined, depends on parameters a, b, c, q, N only, and is supported on $N+1$ points:

$$\{x_k = \left(q^k a + q^{-k} a^{-1}\right)/2 : k = 0, \dots, N\}\ ,$$

According to [3], the Askey-Wilson law assigns to x_k the probability $p_{k,N}(a,b,c) = p_k(a,b,c,1/(q^n a))$ (recall (3.6)). The formula simplifies to

(4.1)
$$p_{k,N}(a,b,c) = \begin{bmatrix} N \\ k \end{bmatrix} \frac{\left(\frac{q^{k+1}a}{b},\ \frac{q^{k+1}a}{c}\right)_{N-k} (ab,\ ac)_k (1 - q^{2k}a^2) q^{\frac{k(k+1)}{2}}}{(q^k a^2)_{N+1} \left(\frac{q}{bc}\right)_N (-bc)^k},\ k = 0, \dots, N\ .$$

Here $\begin{bmatrix} N \\ k \end{bmatrix} = \frac{(q)_N}{(q)_k (q)_{N-k}}$ denotes the q-binomial coefficient.

We remark that if X is a random variable distributed according to $\nu(dx; a, b, c, 1/(aq^N))$ then $\mathbb{E}(X)$ and $\mathrm{Var}(X)$ are given by formulas (2.6) and (2.7) with $d = 1/(q^n a)$ respectively. This can be seen by a discrete version of the calculations from the proof of Proposition 2.2; alternatively, one can use the fact that the first two Askey-Wilson polynomials $\bar{w}_1(X)$ and $\bar{w}_2(X)$ integrate to zero.

The discrete version of Proposition 2.3 says that with $d = 1/(maq^N)$,

(4.2)
$$\nu(U; a, b, cm, md) = \int \nu(U; a, b, m(x + \sqrt{x^2-1}), m(x - \sqrt{x^2-1}))\nu(dx; ma, mb, c, d),$$

and takes the following form.

Lemma 4.2. *For any $m \in (0,1)$ and any $j = 0, 1, \dots, N$*

(4.3)
$$p_{j,N}(a, b, mc) = \sum_{k=j}^{N} p_{j,k}(a, b, q^k m^2 a) p_{k,N}(ma, mb, c)\ .$$

Proof. Expanding the right hand side of (4.3) we have

(4.4)
$$\sum_{k=j}^{N} \begin{bmatrix} k \\ j \end{bmatrix} \frac{\left(\frac{q^{j+1}a}{b},\ \frac{q}{q^{k-j}m^2}\right)_{k-j}}{(q^j a^2)_{k+1}} \frac{(ab,\ q^k m^2 a^2)_j}{\left(\frac{q}{q^k m^2 ab}\right)_k} \frac{(1 - q^{2j}a^2) q^{\frac{j(j+1)}{2}}}{(-q^k m^2 ab)^j}$$
$$\times \begin{bmatrix} N \\ k \end{bmatrix} \frac{\left(\frac{q^{k+1}a}{b},\ \frac{q^{k+1}ma}{c}\right)_{N-k}}{(q^k m^2 a^2)_{N+1}} \frac{(m^2 ab,\ mac)_k}{\left(\frac{q}{mbc}\right)_N} \frac{(1 - q^{2k}m^2 a^2) q^{\frac{k(k+1)}{2}}}{(-mbc)^k}$$
$$= \begin{bmatrix} N \\ j \end{bmatrix} \frac{\left(\frac{q^{j+1}a}{b},\ \frac{q^{j+1}a}{mc}\right)_{N-j} (ab,\ mac)_j (1 - a^2 q^{2j}) q^{\frac{j(j+1)}{2}}}{(q^j a^2)_{N+1} \left(\frac{q}{mbc}\right)_N (-mcb)^j}$$
$$\sum_{k=j}^{N} \begin{bmatrix} N-j \\ k-j \end{bmatrix} \frac{\left(\frac{q^{k+1}ma}{c}, q^{k+1}a^2\right)_{N-k} (m^2)_{k-j} (mq^j ac)_{k-j} (1 - q^{2k}m^2 a^2) q^{\frac{(k-j)(k-j+1)}{2}}}{(q^{k+j}m^2 a^2)_{N-j+1} \left(\frac{q^{j+1}a}{mc}\right)_{N-j} \left(-\frac{mc}{q^j a}\right)^{k-j}}.$$

Here we used identities (2.1) and (2.2). The first one for: (i) $\alpha = q^{j+1}a/b$, $M = k-j$ and $L = N-k$; (ii) $\alpha = q^k m^2 a^2$, $M = j$, $L = N-j+1$; (iii) $\alpha = q^j a^2$, $M = k+1$, $L = N-k$. The second one for: (i) $\alpha = m^2 ab$ and $M = k$; (ii) $\alpha = m^2$ and $M = k-j$.

We transform the sum in (4.4) introducing $K = k-j$, $L = N-j$, $\alpha = mq^j a$, $\beta = \frac{m}{q^j a}$ and $\gamma = c$. Then by (4.1) we get

$$\sum_{K=0}^{L} \begin{bmatrix} L \\ K \end{bmatrix} \frac{\left(\frac{q^{K+1}\alpha}{\gamma},\ q^{K+1}\frac{\alpha}{\beta}\right)_{L-K} (\alpha\beta)_{k-j}(\alpha\gamma)_K}{(q^K\alpha^2)_{L+1}\left(\frac{q}{\beta\gamma}\right)_L} \frac{(1-q^{2K}\alpha^2)q^{\frac{K(K+1)}{2}}}{(-\beta\gamma)^K} = 1 \,.$$

Now the result follows since the first part of the expression (4.4) is the desired probability mass function.

□

4.2. **Markov processes with discrete Askey-Wilson laws.** We now choose the parameters as in Proposition 4.1: $0 < A < B$, $0 < C < D = 1/(Aq^N)$, $ABCD < 1$, $BC < 1$, and choose the time interval $I = \left(C(q^N A)^{-1}, (AB)^{-1}\right)$ from (1.19). For any $t \in I$, define the discrete distribution $\pi_t(dx) = \sum_{k=0}^N \pi_t(y_k(t))\delta_{y_k(t)}(dx)$ by choosing the support from (3.9) with weights

$$\pi_t(y_k(t)) = p_{k,N}\left(As^{\frac12}, Bt^{\frac12}, Ct^{-\frac12}\right) \,. \tag{4.5}$$

Clearly, the support of π_s is $U_s = \{y_0(s), y_1(s), \dots, y_N(s)\}$.
Also for any $s,t \in I$, $s < t$ and for any $k \in \{0,1,\dots,N\}$, define the discrete Askey-Wilson distribution $P_{s,t}(y_k(s), dx) = \sum_{j=0}^k P_{s,t,y_k(s)}\delta_{y_j(t)}(dy)$ by

$$P_{s,t,y_k(s)}(y_j(t)) = p_{j,k}\left(At^{\frac12}, Bt^{\frac12}, q^k Ast^{-\frac12}\right) \,. \tag{4.6}$$

Thus $P_{s,t}(x, dy)$ is defined only for x from the support of π_s. Next, we give the discrete version of Proposition 2.4.

Proposition 4.3. *The family of distributions $(\pi_t, P_{s,t}(x,dy))$, $s,t \in I$, $s < t$, $k \in \{0,1,\dots,N\}$ defines a Markov process $(Y_t)_{t\in I}$ with trajectories contained in the set of functions $\{(y_k(t))_{t\in I},\ k = 0,1,\dots,N\}$.*

Proof. We need to check the Chapman-Kolmogorov conditions. Note that for any $s < t$ and any k the support of the measure $P_{s,t,y_k(s)}$ is a subset of the support U_t of the measures π_t. First we check

$$\pi_t(y_j(t)) = \sum_{k=j}^{N} P_{s,t,k}(y_j(t))\,\pi_s(y_k(s)) \tag{4.7}$$

which can be written as

$$p_{j,N}\left(At^{\frac12}, Bt^{\frac12}, Ct^{-\frac12}\right) = \sum_{k=j}^{N} p_{j,k}\left(At^{\frac12}, Bt^{\frac12}, q^k Ast^{-\frac12}\right) p_{k,N}\left(As^{\frac12}, Bs^{\frac12}, Cs^{-\frac12}\right) \,.$$

Now (4.7) follows from (4.3) with $a = At^{\frac12}$, $b = Bt^{\frac12}$, $c = Cs^{-\frac12}$ and $m = (s/t)^{\frac12}$.

Similarly, the condition

$$P_{s,u,u_k(s)}(u_i(u)) = \sum_{j=k}^{i} P_{t,u,u_j(t)}(u_i(u))\, P_{s,t,u_k(s)}(u_j(t)) \tag{4.8}$$

assumes the form

$$p_{i,k}(Au^{\frac{1}{2}}, Bu^{\frac{1}{2}}, q^k Asu^{-\frac{1}{2}}) \\ = \sum_{j=k}^{i} p_{i,j}(Au^{\frac{1}{2}}, Bu^{\frac{1}{2}}, q^j Atu^{-\frac{1}{2}}) p_{j,k}(At^{\frac{1}{2}}, Bt^{\frac{1}{2}}, q^k Ast^{-\frac{1}{2}}) \, .$$

Therefore (4.8) follows from (4.3) with $(j,k,N) \to (i,j,k)$, $a = Au^{\frac{1}{2}}$, $b = Bu^{\frac{1}{2}}$, $c = q^k Ast^{-\frac{1}{2}}$ and $m = (t/u)^{\frac{1}{2}}$. $\square$

Let $(Y_t)_{t\in I}$ be a Markov process defined by the above Markov family $(\pi_t, P_{s,t,y_k(s)})$.

Note that at the end-points of I, $Y_{C/(q^N A)}$ is degenerate at $\frac{1+ACq^N}{2(q^N AC)^{\frac{1}{2}}}$ and $Y_{1/(AB)}$ is degenerate at $\frac{A+B}{2(AB)^{\frac{1}{2}}}$ (compare Proposition 5.1(i)).

Proposition 4.4. *For the process $(Y_t)_{t\in I}$ defined above*

$$\mathbb{E}(Y_t) = \frac{(1-q^N)A(Bt+C) - (1-q^N A^2 t)(1-BC)}{2At^{\frac{1}{2}}(BC-q^N)} \, ,$$

$$\mathrm{Cov}(Y_s, Y_t) \\ = \frac{(1-q)(1-q^N)(1-AC)(1-BC)(q^n A - B)(q^N As - C)(1-ABt)}{4q(st)^{\frac{1}{2}}(BC-q^N)^2(BC-q^{N-1})} \, ,$$

$$\mathrm{Var}(Y_t) = \frac{(1-AC)(1-BC)(1-q^N)(1-q)(1-ABt)(C-q^N At)}{4qAt(BC-q^N)^2(BC-q^{N-1})} \, .$$

Expressions for conditional expectations and conditional variances are exactly the same as in the absolutely continuous case.

Proof. The result follows from the fact that the marginal and conditional distributions of the process $(Y_t)_{t\in I}$ are finite Askey-Wilson. Therefore one can apply the formulas (2.6) and (2.7). The covariance is derived through conditioning $\mathbb{E}(Y_s Y_t) = \mathbb{E}(Y_s \mathbb{E}(Y_t|\mathcal{F}_s))$. $\square$

Note that the above formulas are identical to those in the case with densities by plugging there $D = q^{-N} A^{-1}$.

Since we are interested in the harness properties, we want to find the conditional distributions of the process with conditioning with respect to the past and the future, jointly. The following result says that the conditional distribution of Y_t given the (admissible) values $Y_s = x, Y_u = z$ is a discrete Askey-Wilson distribution

$$\nu\left(dy; \sqrt{\frac{t}{u}}(z+\sqrt{z^2-1}), \sqrt{\frac{t}{u}}(z-\sqrt{z^2-1}), \sqrt{\frac{s}{t}}(x+\sqrt{x^2-1}), \frac{s}{t}(x-\sqrt{x^2-1})\right),$$

compare Proposition 2.6. (Using notation (4.1), this formula takes a much more concise form.)

Proposition 4.5. *Let $(Y_t)_{t\in I}$ be the Markov process defined by $(\pi_t, P_{s,t,y_k(s)})$ given by (4.5) and (4.6) with parameters A, B, C, q, N. Then for any $s, t, u \in I$ such that*

$s < t < u$ the conditional distribution of Y_t given $\mathcal{F}_{s,u}$ is defined by the finite Askey-Wilson distribution

$$P(Y_t = y_j(t)|Y_s = y_k(s), Y_u = y_i(u)) = p_{j-i,k-i}\left(q^i A t^{\frac12}, \frac{t^{\frac12}}{q^i A u}, \frac{q^k A s}{t^{\frac12}}\right).$$

The expressions for the first two conditional moments are the same as in the absolutely continuous case.

Proof. Due to the Markov property of the process $(Y_t)_{t\in I}$, to determine the conditional distribution of Y_t given $\mathcal{F}_{s,u}$ it suffices to find $P(Y_t = y_j(t)|Y_s = y_k(s), Y_u = y_i(u))$ for any $i, j, k \in \{0, 1, \ldots, N\}$, such that $i \le j \le k$. Also the Markov property implies that this probability can be expressed in terms of conditional probabilities with respect to the past as

$$p(j|k,i) = P(Y_t = y_j(t)|Y_s = y_k(s), Y_u = y_i(u))$$

$$= \frac{P(Y_u = y_i(u)|Y_t = y_j(t))P(Y_t = y_j(t)|Y_s = y_k(s))}{P(Y_u = y_i(u)|Y_s = y_k(s))}$$

$$= \frac{p_{i,j}(Au^{\frac12}, Bu^{\frac12}, q^j A t u^{-\frac12})\, p_{j,k}(At^{\frac12}, Bt^{\frac12}, q^k A s t^{-\frac12})}{p_{i,k}(Au^{\frac12}, Bu^{\frac12}, q^k A s u^{-\frac12})}$$

Expanding the expression for the probability mass functions according to (4.1) we get

$$p(j|k,i) = \begin{bmatrix} j \\ i \end{bmatrix} \frac{\left(\frac{q^{i+1}A}{B},\ \frac{qu/t}{q^{j-i}}\right)_{j-i}}{(q^i A^2 u)_{j+1}} \frac{(ABu,\ q^j A^2 t)_i}{\left(\frac{q}{q^j ABt}\right)_j} \frac{(1-q^{2i}A^2u)q^{\frac{i(i+1)}{2}}}{(-q^j ABt)^i}$$

$$\begin{bmatrix} k \\ j \end{bmatrix} \frac{\left(\frac{q^{j+1}A}{B},\ \frac{qt/s}{q^{k-j}}\right)_{k-j}}{(q^j A^2 t)_{k+1}} \frac{(ABt,\ q^k A^2 s)_j}{\left(\frac{q}{q^k ABs}\right)_k} \frac{(1-q^{2j}A^2t)q^{\frac{j(j+1)}{2}}}{(-q^k ABs)^j}$$

$$\left\{\begin{bmatrix} k \\ i \end{bmatrix} \frac{\left(\frac{q^{i+1}A}{B},\ \frac{qu/s}{q^{k-i}}\right)_{k-i}}{(q^i A^2 u)_{k+1}} \frac{(ABu,\ q^k A^2 s)_i}{\left(\frac{q}{q^k ABs}\right)_k} \frac{(1-q^{2i}A^2u)q^{\frac{i(i+1)}{2}}}{(-q^k ABs)^i}\right\}^{-1}$$

This can be reduced in several steps. The q-binomial symbols reduce, as in the classical ($q = 1$) case to $\begin{bmatrix} k-i \\ j-i \end{bmatrix}$. Then we apply (2.1) in the following situations: (i) $\alpha = \frac{q^{i+1}A}{B}$, $M = j-i$, $L = k-j$, (ii) $\alpha = q^i A^2 u$, $M = j+1$, $L = k-j$, (iii) $\alpha = q^j A^2 t$, $M = i$, $L = k-i+1$, (iv) $\alpha = q^k A^2 s$, $M = i$, $L = j-i$. Also we apply (2.2) for (i) $\alpha = \frac{u}{t}$, $M = j-i$, (ii) $\alpha = ABt$, $M = j$. Thus

$$p(j|k,i) = \begin{bmatrix} k-i \\ j-i \end{bmatrix} \frac{\left(\frac{qt/s}{q^{k-j}},\ q^{i+j+1}A^2u\right)_{k-j} \left(\frac{t}{u},\ q^{k+i}A^2s\right)_{j-i} (1-q^{2j}A^2t)q^{\frac{(j-i)(j-i+1)}{2}}}{(q^{i+j}A^2t)_{k-i+1} \left(\frac{q}{q^{k-i}\frac{u}{s}}\right)_{k-i} \left(-q^{k-i}\frac{s}{u}\right)^{j-i}}$$

which, through comparison with the definition (4.1), is easily identified as the distribution we sought.

∎

Proof of Proposition 4.1. Since formulas (2.6) and (2.7) hold for all Askey-Wilson distributions, from Proposition 4.5 we see that the conditional moments and variances in the discrete case are also given by formulas from Proposition 2.5. Therefore the transformed process

$$X_t = \frac{(1+ABt)2T(t)^{\frac{1}{2}}Y_{T(t)} - (A+B)t - (C+\frac{1}{q^N A})}{\sqrt{(1-q)(1-AC)(1-q^{-N})(1-BC)(1-\frac{B}{q^N A})}}\sqrt{1-q^{-N+1}BC}\ , t\in J$$

is a quadratic harness on J with $\theta,\eta,\tau,\sigma,\gamma$ defined as in the general case with $D=q^{-N}A^{-N}$. $\square$

5. Concluding observations

This section contains additional observations that may merit further study.

5.1. **Bridge property.** The following proposition lists some combinations of parameters that create a "quadratic harness bridge" either between two-point masses, or degenerated laws.

Proposition 5.1. *Let $(Z_t)_{t\in I}$ be the Markov process from Theorem 1.2. Assume that $AB\neq 0$ so that (1.19) defines a finite interval $I=[S_0,S_1]$, and extend Z_t to the end-points of I by L_2-continuity.*

(i) *If $AB>0$ then $Z_{S_2}=(1/A+1/B)/\sqrt{1-q}$ is deterministic; similarly, if $CD\geq 0$ then $Z_{S_1}=(C+D)/\sqrt{1-q}$.*
(ii) *If $q\leq 0$ and $CD<0$ then Z_{S_1} takes only two-values. Similarly, if $q\leq 0$ and $AB<0$ then Z_{S_2} is a two-valued random variable.*
(iii) *If $CD<0$ and $q>0$ then Z_0 is purely discrete with the following law:*

$$\Pr\left(Z_0=\frac{q^kC}{\sqrt{1-q}}\right) = \frac{(AD,BD)_\infty(AC,BC)_k}{(D/C,ABCD)_\infty(q,qC/D)_k}q^k,\ k\geq 0\,, \tag{5.1}$$

$$\Pr\left(Z_0=\frac{q^kD}{\sqrt{1-q}}\right) = \frac{(AC,BC)_\infty(AD,BD)_k}{(C/D,ABCD)_\infty(q,qD/C)_k}q^k,\ k\geq 0\,. \tag{5.2}$$

Proof. We can derive the first two statements from moments, which are easier to compute for (Y_t) instead of (Z_t). In the first case, $\mathrm{Var}(Y_t)=0$ at the endpoints, see (2.15); in the second case $E(\bar w_2^2(Y_t))=0$ at the end-points. Alternatively, one can compute the limit of the Askey-Wilson law as in the proof of part (iii).

For part (iii), without loss of generality assume $|A|\leq|B|$ and $|C|\leq|D|$. Then the discrete part of Z_s has atoms at $\left\{\frac{1}{\sqrt{1-q}}\left(q^jC+\frac{s}{Cq^j}\right):\ j\geq 0,\ q^{2j}C^2>s\right\}$ and $\left\{\frac{1}{\sqrt{1-q}}\left(q^jD+\frac{s}{Dq^j}\right):\ j\geq 0,\ q^{2k}D^2>s\right\}$.

The probabilities can be computed from (3.6) with $c=A\sqrt{s}, d=B\sqrt{s}$ and either $a=C/\sqrt{s}, b=D/\sqrt{s}$ for (5.1) or $a=D/\sqrt{s}, b=C/\sqrt{s}$ for (5.2), and converge to (5.1) and (5.2) respectively. To see that the limit distribution is indeed discrete, we note that

$$\sum_{k=0}^\infty \Pr\left(Z_0=\frac{q^kC}{\sqrt{1-q}}\right)+\Pr\left(Z_0=\frac{q^kC}{\sqrt{1-q}}\right)$$
$$=\frac{(AD,BD)_\infty}{(D/C,ABCD)_\infty}\ {}_2\varphi_1\left(\begin{matrix}AC,BC\\ qC/D\end{matrix};q\right)+\frac{(AC,BC)_\infty}{(C/D,ABCD)_\infty}\ {}_2\varphi_1\left(\begin{matrix}AD,BD\\ qD/C\end{matrix};q\right)=1.$$

Here we use hypergeometric function notation

$$ {}_{r+1}\varphi_r \left(\begin{matrix} a_1, a_2, \dots, a_{r+1} \\ b_1, b_2, \dots, b_r \end{matrix} ; z \right) = \sum_{k=0}^{\infty} \frac{(a_1, a_2, \dots, a_r)_k}{(q, b_1, \dots, b_r)_k} z^k. $$

The identity that gives the final equality is [16, (12.2.21)] used with $a = AC, b = BC, c = qC/D$. $\square$

5.2. **Transformations that preserve quadratic harness property.** The basic idea behind the transformation (2.27) is that if covariance $\mathbb{E}(Z_t Z_s) = c_0 + c_1 \min\{t, s\} + c_2 \max\{t, s\} + c_3 ts$ factors as $(s - \alpha)(1 - t\beta)$ for $s < t$ with $\alpha\beta < 1$ then it can be transformed into $\min\{t, s\}$ by a deterministic time change and scaling.

This transformation is based on the following group action: if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(\mathbb{R})$ is invertible, then A acts on stochastic processes $\mathbf{X} = (X_t)$ by $A(\mathbf{X}) := \mathbf{Y}$ with $Y_t = (ct + d) X_{T_A(t)}$, where $T_A(t) = (at + b)/(ct + d)$ is the associated Möbius transformation. It is easy to check that this is a (right) group action: $A(B(\mathbf{X})) = (B \times A)(\mathbf{X})$.

If $\mathbb{E}(Z_t Z_s) = (s - \alpha)(1 - t\beta)$ for $s < t$ and $\alpha\beta < 1$ then $\mathbf{X} = A^{-1}(\mathbf{Z})$ with $A = \begin{bmatrix} 1 & -\alpha \\ -\beta & 1 \end{bmatrix}$ has $\mathbb{E}(X_t X_s) = \min\{s, t\}$. The easiest way to see this is to note that T_A is increasing for $\alpha\beta < 1$, and by group property $\mathbf{Z} = A(\mathbf{X})$. So with $s < t$, $\mathbb{E}(Z_s Z_t) = (1 - s\beta)(1 - t\beta)\mathbb{E}\left(X_{T_A(s)} X_{T_A(t)}\right) = (1 - s\beta)(1 - t\beta) T_A(s) = (s - \alpha)(1 - t\beta)$.

It is clear that, at least locally, this group action preserves properties of linearity of regression and of quadratic conditional variance. In fact, one can verify that the general form of the covariance $\mathbb{E}(X_t, X_s) = c_0 + c_1 \min\{t, s\} + c_2 \max\{t, s\} + c_3 ts$ is also preserved, and since this covariance corresponds to (1.2), the latter is also preserved by the group action.

Acknowledgement. The authors thank Persi Diaconis, Mourad Ismail, Wojciech Matysiak, and Ryszard Szwarc for helpful discussions and valuable information. This research was partially supported by Taft Research Seminar 2008-2009.

References

[1] M. Anshelevich. Free martingale polynomials. *J. Funct. Anal.*, 201:228–261, 2003. ArXiv:math.CO/0112194.

[2] M. Anshelevich. Appell polynomials and their relatives. *Int. Math. Res. Not.*, (65):3469–3531, 2004.

[3] R. Askey and J. Wilson. A set of orthogonal polynomials that generalize the Racah coefficients or $6 - j$ symbols. *SIAM J. Math. Anal.*, 10(5):1008–1016, 1979.

[4] R. Askey and J. Wilson. Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials. *Mem. Amer. Math. Soc.*, 54(319):iv+55, 1985.

[5] D. Bakry and O. Mazet. Characterization of Markov semigroups on $\mathbb{R}$ associated to some families of orthogonal polynomials. In J. Azéma, M. Émery, M. Ledoux, and M. Yor, editors, *Séminaire de Probabilités XXXVII*, volume 1832 of *Lecture Notes in Mathematics*, pages 60 – 80. Springer-Verlag, Jan 2003.

[6] P. Biane. Processes with free increments. *Math. Z.*, 227(1):143–174, 1998.

[7] M. Bożejko and W. Bryc. On a class of free Lévy laws related to a regression problem. *J. Funct. Anal.*, 236:59–77, 2006. arxiv.org/abs/math.OA/0410601.

[8] W. Bryc, W. Matysiak, and J. Wesołowski. Quadratic harnesses, q-commutations, and orthogonal martingale polynomials. *Trans. Amer. Math. Soc.*, 359:5449–5483, 2007. arxiv.org/abs/math.PR/0504194.

[9] W. Bryc, W. Matysiak, and J. Wesołowski. The bi-Poisson process: a quadratic harness. *Ann. Probab.*, 36:623–646, 2008. arxiv.org/abs/math.PR/0510208.

[10] W. Bryc and J. Wesołowski. Conditional moments of q-Meixner processes. *Probab. Theory Related Fields*, 131:415–441, 2005. arxiv.org/abs/math.PR/0403016.

[11] W. Bryc and J. Wesołowski. Bi-Poisson process. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, 10:277–291, 2007. arxiv.org/abs/math.PR/0404241.

[12] P. Diaconis, K. Khare, and L. Saloff-Coste. Rejoiner: Gibbs sampling, exponential families and orthogonal polynomials. *Statistical Science*, 23:196–200, 2008.

[13] P. Feinsilver. Some classes of orthogonal polynomials associated with martingales. *Proc. Amer. Math. Soc.*, 98(2):298–302, 1986.

[14] J. M. Hammersley. Harnesses. In *Proc. Fifth Berkeley Sympos. Mathematical Statistics and Probability (Berkeley, Calif., 1965/66), Vol. III: Physical Sciences*, pages 89–117. Univ. California Press, Berkeley, Calif., 1967.

[15] F. Hiai and D. Petz. *The semicircle law, free random variables and entropy*, volume 77 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2000.

[16] M. E. H. Ismail. *Classical and quantum orthogonal polynomials in one variable*, volume 98 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2005. With two chapters by Walter Van Assche, With a foreword by Richard A. Askey.

[17] R. Koekoek and R. F. Swarttouw. The askey scheme of hypergeometric orthogonal polynomials and its q-analogue, 1998. Delft University of Technology Report no. 98-17, http://fa.its.tudelft.nl/∼koekoek/askey.html.

[18] E. Lytvynov. Polynomials of Meixner's type in infinite dimensions—Jacobi fields and orthogonality measures. *J. Funct. Anal.*, 200(1):118–149, 2003.

[19] R. Mansuy and M. Yor. Harnesses, Lévy bridges and Monsieur Jourdain. *Stochastic Processes and Their Applications*, 115:329–338, February 2005.

[20] B. Nassrallah and M. Rahman. Projection formulas, a reproducing kernel and a generating function for q-Wilson polynomials. *SIAM J. Math. Anal.*, 16(1):186–197, 1985.

[21] M. Noumi and J. V. Stokman. Askey-Wilson polynomials: an affine Hecke algebra approach. In *Laredo Lectures on Orthogonal Polynomials and Special Functions*, Adv. Theory Spec. Funct. Orthogonal Polynomials, pages 111–144. Nova Sci. Publ., Hauppauge, NY, 2004.

[22] D. Nualart and W. Schoutens. Chaotic and predictable representations for Lévy processes. *Stochastic Process. Appl.*, 90(1):109–122, 2000.

[23] N. Saitoh and H. Yoshida. A q-deformed Poisson distribution based on orthogonal polynomials. *J. Phys. A*, 33(7):1435–1444, 2000.

[24] W. Schoutens. *Stochastic processes and orthogonal polynomials*, volume 146 of *Lecture Notes in Statistics*. Springer-Verlag, New York, 2000.

[25] W. Schoutens and J. L. Teugels. Lévy processes, polynomials and martingales. *Comm. Statist. Stochastic Models*, 14(1-2):335–349, 1998. Special issue in honor of Marcel F. Neuts.

[26] D. B. Sears. On the transformation theory of basic hypergeometric functions. *Proc. London Math. Soc. (2)*, 53:158–180, 1951.

[27] J. L. Solé and F. Utzet. On the orthogonal polynomials associated with a Lévy process. *Ann. Probab.*, 36(2):765–795, 2008.

[28] J. L. Solé and F. Utzet. Time-space harmonic polynomials relative to a Lévy process. *Bernoulli*, 14(1):1–13, 2008.

[29] J. V. Stokman and T. H. Koornwinder. On some limit cases of Askey-Wilson polynomials. *J. Approx. Theory*, 95(2):310–330, 1998.

[30] M. Uchiyama, T. Sasamoto, and M. Wadati. Asymmetric simple exclusion process with open boundaries and Askey-Wilson polynomials. *J. Phys. A*, 37(18):4985–5002, 2004.

[31] J. Wesołowski. Stochastic processes with linear conditional expectation and quadratic conditional variance. *Probab. Math. Statist.*, 14:33–44, 1993.

[32] D. Williams. Some basic theorems on harnesses. In *Stochastic analysis (a tribute to the memory of Rollo Davidson)*, pages 349–363. Wiley, London, 1973.

Department of Mathematics, University of Cincinnati, PO Box 210025, Cincinnati, OH 45221–0025, USA
E-mail address: Wlodzimierz.Bryc@UC.edu

Faculty of Mathematics and Information Science, Warsaw University of Technology, pl. Politechniki 1, 00-661 Warszawa, Poland
E-mail address: wesolo@alpha.mini.pw.edu.pl